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FIXED POINT THEOREM FOR ĆIRIĆ'S QUASICONTRACTION IN GENERALIZED COMPLEX VALUED METRIC SPACES

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ABSTRACT. In this work, we introduce some properties of the generalized complex valued metric space and extend some fixed point results, that is Ćirić's fixed point theorem. Some are recover various complex valued metric space and complex valued b-metric space. Our results extend and improve some results of Mohamed Jleli and Bessem Samet [17].

KEYWORDS: generalized complex valued metric space, Ćirić's quasicontraction.
AMS Subject Classification: 47H10; 54H25; 30L15

1. Introduction

The study of fixed point theorems has attracted considerable attention from many mathematicians. In 1906, Fréchet introduced fixed point theorems in metric spaces [13]. Later, in 1922, Banach presented a fixed point theorem for contraction mappings in metric spaces [6].

In recent years, numerous researchers have proven fixed point theorems in generalized metric spaces; for examples, see [10, 2, 22] and the references therein. The concept of dislocated metric spaces was introduced in 2000 by Hitzler and Seda [15]; see [1]-[19] and references therein.

Very recently, A. Azam, B. Fisher, and M. Khan [4] introduced the concept of complex-valued metric spaces and established common fixed point theorems for a pair of mappings that satisfy a contractive condition in these spaces.

Recently, Jleli and Samet [18] introduced a new concept of generalized metric spaces, extending several well-known fixed point results, including the Banach contraction principle. In 2017, Elkouch and Marhrani [12] established existence results for the Kannan contraction in generalized metric spaces.

n this paper, inspired by the work of Elkouch and Marhrani [12], we introduce a generalized complex-valued metric space. We explore the relationships between

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this space and complex-valued b-metric space, complex-valued dislocated metric spaces, and complex-valued metric spaces. In the final section, we prove a fixed point theorem for a mapping T satisfying the Ćirić's k-quasicontraction condition.

2. Preliminaries

In this section, we give some definitions and lemmas for this work.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a **metric space**, and d is called a metric on X.

In 2000, Hitzler and Seda [15], introduced the notion of dislocated metric space as follows.

Definition 2.2. [15] Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a dislocated metric on X if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0 \quad \Rightarrow \quad x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a **dislocated metric space**.

It is easy to show that, the metric space X is dislocated metric space.

Next, we suppose the definition of b-metric space, this space is more generalized than metric spaces.

Definition 2.3. [5] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a **b-metric space**. The number $s \geq 1$ is called the coefficient of (X, d).

The following is an example of b-metric spaces.

Example 2.4. [5] Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

In 2017, Elkouch and Marhrani [12] defined a new class of metric space, let X be a nonempty set, and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to \infty} D(x_n, x) = 0 \right\}.$$

Definition 2.5. ([12]) A mapping D is called a generalized metric if it satisfies the following conditions:
1. For every $(x, y) \in X \times X$, we have

$$D(x, y) = 0 \Rightarrow x = y.$$

2. For every $(x, y) \in X \times X$, we have

$$D(x, y) = D(y, x).$$

3. There exists a real constant $C > 0$ such that for all $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$, we have

$$D(x, y) \leq C \limsup_{n \to \infty} D(x_n, y).$$

The pair (X, D) is called a **generalized metric space**.

It is not difficult to observe that metric d in Definition 2.1 satisfies all the conditions $(i) - (iii)$ with $C = 1$. In 2015 Mohamed Jleli and Bessem Samet [17] proved that any dislocated metric space is a generalized metric and any b-metric on X is a generalized metric on X.

In this work, we will study the generalized metric space in a complex form. Let $\mathbf{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbf{C}$. Define a partial order relation $\preceq$ on $\mathbf{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \preceq z_2$ if one of the following holds:
(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precneq z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$, i.e., one of (2),(3) and (4) is satisfied and we will write $z_1 \prec z_2$ only (4) is satisfied.

Remark 2.6. We can easily check the following:
(i) If $a, b \in \mathbf{R}, 0 \leq a \leq b$ and $z_1 \preceq z_2$ then $az_1 \preceq bz_2, \forall z_1, z_2 \in \mathbf{C}$.
(ii) $0 \preceq z_1 \precneq z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \preceq z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.
Azam et al. [4] defined the complex valued metric space in the following way:

Lemma 2.7. *For any $z \in \mathbf{C}$ with $0 \prec z$, there exists $r \in \mathbf{C}$ with $0 \prec r$ such that $z = r|z|$.*

Proof Let $z \in \mathbf{C}$ with $0 \prec z$. Put $r = \frac{Re(z)}{|z|} + \frac{Im(z)}{|z|}i \succ 0$. It implied that

$$\begin{aligned} z &= Re(z) + Im(z)i \\ &= \frac{Re(z)}{|z|} \cdot |z| + \frac{Im(z)}{|z|} i \cdot |z| \\ &= \left[\frac{Re(z)}{|z|} + \frac{Im(z)}{|z|} i\right] |z| \\ &= r \cdot |z| \end{aligned}$$

This complete the proof.

[4] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbf{C}$ satisfies the following conditions:
$(C1)$ $0 \preceq d(x, y)$, for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
$(C2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(C3)$ $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y \in X$.

Then d is called a complex valued metric on X and (X, d) is called a **complex valued metric space**.

Definition 2.8. Let X be a nonempty set. A function $d : X \times X \to \mathbf{C}$ is called a complex valued dislocated metric on X if for $x, y, z \in X$ the following conditions are satisfied:
(i) $d(x, y) = 0 \quad \Rightarrow \quad x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \preceq d(x, y) + d(y, z)$.
The pair (X, d) is called a **complex valued dislocated metric space**.

Definition 2.9. [24] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbf{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in \mathbf{C}$, the following conditions are satisfied:
(i) $0 \preceq d(x, y)$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, z) \preceq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space which is defined in Definition **??**. The following example is an example of complex valued b-metric space.

Example 2.10. [24] Let $X = \mathbf{C}$. Define the mapping $d : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbf{C}, d)$ is complex valued b-metriic space with $s = 2$.

In this work, we consider a nonempty set X, and $D : X \times X \to \mathbf{C}$ be a given mapping. For every $x \in X$, we define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to \infty} |D(x_n, x)| = 0 \right\}.$$

Definition 2.11. Let X be a nonempty set, a mapping $D : X \times X \to \mathbf{C}$ is called a generalized complex value metric if it satisfies the following conditions:
1. For every $x, y \in X$, we have

$$0 \preceq D(x, y).$$

2. For every $x, y \in X$, we have

$$D(x, y) = 0 \Rightarrow x = y.$$

3. For all $x, y \in X$, we have

$$D(x, y) = D(y, x).$$

4. There exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and $\{x_n\} \in C(D, X, x)$, we have

$$D(x, y) \preceq r \limsup_{n \to \infty} |D(x_n, y)|.$$

Then a pair (X, D) is called a **generalized complex valued metric space**.

Definition 2.12. [12] Let (X, D) be a generalized complex valued metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. We say that $\{x_n\}$ is converge to x in X, if $\{x_n\} \in C(D, X, x)$. We denote by $\lim_{n \to \infty} x_n = x$.

Example 2.13. [16] Let $X = [0, 1]$ and $D : X \times X \to \mathbf{C}$ be the mapping defined by for any $x, y \in X$

$$\begin{cases} D(x, y) = (x + y)i; x \neq 0 \text{ and } y \neq 0 \\ D(x, 0) = D(0, x) = \frac{x}{2}i. \end{cases}$$

Proof Let $x, y \in X$, we have $x \geq 0$ and $y \geq 0$, thus $x + y \geq 0$.

If $D(x, y) = (x + y)i = 0 + (x + y)i \succeq 0 + 0i = 0$.

If $D(x, 0) = \frac{x}{2}i = 0 + \frac{x}{2}i \succeq 0 + 0i = 0$.

$$\text{Hence, } D(x, y) \succeq 0.$$

If $D(x, y) = 0$, then $(x + y)i = 0$. Hence, $x = 0 = y$.

If $x \neq 0$ and $y \neq 0$, $D(x, y) = (x+y)i = (y+x)i = D(y, x)$ and $D(x, 0) = D(0, x)$.

Let $\{x_n\} = \{\frac{(n-1)x}{n}\} \subseteq X$, we see that $\limsup_{n\to\infty} |D(x_n, x)| = 0$ and put $r = i$, then we have

$$D(0, y) = \frac{y}{2}i \text{ and } \limsup_{n\to\infty} |D(x_n, y)| = \limsup_{n\to\infty} \sqrt{(\frac{(n-1)x}{n} + y)^2} = x + y.$$

Hence, $D(0, y) = \frac{y}{2}i \preceq (x + y)i$, and we see that

$$D(x, y) = (x + y)i \text{ and } \limsup_{n\to\infty} |D(x_n, y)| = \limsup_{n\to\infty} \sqrt{(\frac{(n-1)x}{n} + y)^2} = x + y.$$

Hence, $D(x, y) = (x + y)i \preceq r \limsup_{n\to\infty} |D(x_n, y)|$.

Definition 2.14. [12] Let (X, D) be a generalized complex valued metric space. Then a sequence $\{x_n\}$ in X is said to Cauchy sequence in X, if $\lim_{n\to\infty} |D(x_n, x_{n+m})| = 0$.

Definition 2.15. [12] Let (X, D) be a generalized complex valued metric space. If every Cauchy sequence is convergent in X then (X, D) is called a complete complex valued metric space.

Definition 2.16. [21] The max function for complex numbers with partial order relation $\preceq$ is defined as

(i) $\max\{z_1, z_2\} = z_2 \Rightarrow z_1 \preceq z_2$;

(ii) $z_1 \preceq \max\{z_1, z_2\} \Rightarrow z_1 \preceq z_2$ or $z_1 \preceq z_3$.

On the similar lines Singh et al. [23] defined min function as

(i) $\min\{z_1, z_2\} = z_1 \Rightarrow z_1 \preceq z_2$;

(ii) $\min\{z_1, z_2\} \preceq z_3 \Rightarrow z_1 \preceq z_3$ or $z_2 \preceq z_3$. Now we introduce the best proximity point and some related concept in complex valued rectangular metric space.

3. Some Property on Generalized Complex Valued Metric Space

In this section, we prove some propositions for use in the main theorem and some fixed point theorems in generalized complex valued metric space.

Proposition 3.1. *Let (X,D) be a generalized complex valued metric space. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.*

Proof Suppose that $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, by Definition 2.12, we have

$$|D(x_n, x)| \to 0, \ |D(x_n, y)| \to 0, \text{ as } n \to \infty.$$

Using the property (4) in Definition 2.11, we have there exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and since $\{x_n\} \in C(D, X, x)$ such that

$$D(x, y) \preceq r \limsup_{n\to\infty} |D(x_n, y)|.$$

Hence, $D(x, y) = 0$. Using property (2) in Definition 2.11, we have $x = y$.

Proposition 3.2. *Any complex valued b-metric space is a generalized complex valued metric space on X.*

Proof Let $\{x_n\} \in C(d, X, x)$. From the Definition 2.9(iv), we have

$$d(x, y) \preceq s[d(x, x_n) + d(x_n, y)].$$

It follows that, from Lemma 2.7, we have there exist $r_1, r_2 \in \mathbf{C}$ with $0 \prec r_1, r_2$ such that

$$\begin{aligned} d(x, x_n) &= r_1|d(x, x_n)| \\ d(x_n, y) &= r_2|d(x_n, y)|. \end{aligned}$$

Then

$$d(x, y) \preceq s[r_1|d(x, x_n)| + r_2|d(x_n, y)|].$$

From $\{x_n\} \in C(d, X, x)$, we have

$$d(x, y) \preceq sr_2 \limsup_{n\to\infty} |d(x_n, y)|].$$

Since $0 \prec r_2$ and $0 \prec s$ then $r = sr_2 \succ 0$ such that

$$d(x, y) \preceq r \limsup_{n\to\infty} |d(x_n, y)|].$$

Hence, (X, d) is a generalized complex valued metric space.

It is not difficult to observe that the complex valued metric d satisfies (1-4) of Definition 2.11 and any complex valued dislocated metric space is generalized complex valued metric space.

4. Ćirić's quasicontraction in Generalized Complex Valued Metric Space

In 1974, Ćirić's [9] introduced a class of self-maps on a metric space (X, d) which satisfy the following condition:

$$d(Sx, Sy) \preceq q \max\{d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\}, \tag{4.1}$$

for every $x, y \in X$ and $0 \leq q < 1$. The maps satisfying Condition 4.1 are said to be quasicontractions.

In this section, we extend Ćirić's fixed point theorem for quasicontraction is a self-maps on generalized complex valued metric space (X, D) defined by:

$$D(Tx, Ty) \preceq k \max\{D(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\},$$

for every $x, y \in X$ and $k \in (0, 1)$. We say that T is a k-quasicontraction

Proposition 4.1. *Suppose that $T : X \to X$ is a k-quasicontraction for some $k \in (0, 1)$. Then any fixed point $p \in X$ of T satisfies*

$$|D(p, p)| < \infty \Rightarrow D(p, p) = 0.$$

Proof Let $p \in X$ be a fixed point of T such that $|D(p, p)| < \infty$. Since T is a k-quasicontraction for some $k \in (0, 1)$, we have

$$\begin{aligned} D(p, p) = D(Tp, Tp) &\preceq k \max\{D(p, p), D(p, Tp), D(p, Tp), D(p, Tp), D(p, Tp)\} \\ &= kD(p, p). \end{aligned}$$

From Remark 2.6(ii), we have

$$|D(p, p)| \leq k|D(p, p)|.$$

Since $k \in (0, 1)$, we get $D(p, p) = 0$. This proof is complete.

Next, we suppose that, for every $x \in X$

$$\delta(D, T, x) = \sup\left\{|D(T^i x, T^j x)| : i, j \in \mathbf{N}\right\}.$$

From Proposition 4.1, we have the following result:

Theorem 4.2. *Let (X,D) be a complete generalized complex valued metric space, and $T : X \to X$ be a k-quasicontraction for some $k \in \Big(0, \inf\{1, \frac{1}{|r|}\}\Big)$ and there exists an element $x_0 \in X$ such that $\delta(D,T,x_0) < \infty$. Then the sequence $\{T^n x_0\}$ converges to some $p \in X$.*

If $D(x_0,Tp) \prec \infty$ and $D(p,Tp) \prec \infty$, then p is a fixed point of T. Moreover, if $p^{'}$ is a fixed point of T in X such that $|D(p,p^{'})| < \infty$ and $|D(p^{'},p^{'})| < \infty$ then $p = p^{'}$.

Proof Let $n \in \mathbf{N}$, for all $i,j \in \mathbf{N}$, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) = D(T(T^{n+i-1}x_0), T(T^{n+j-1}x_0)).$$

By Definition of quasicontraction, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \preceq k \max \left\{ \begin{array}{l} D(T^{n+i-1}x_0, T^{n+j-1}x_0), D(T^{n+i-1}x_0, T^{n+i}x_0), \\ D(T^{n+j-1}x_0, T^{n+j}x_0), D(T^{n+j-1}x_0, T^{n+i}x_0), \\ D(T^{n+i-1}x_0, T^{n+j}x_0) \end{array} \right\}$$

Then we have

$$\delta(D,T,T^n x_0) \leq k\delta(D,T,T^{n-1}x_0).$$

Hence, for any $n \geq 1$, we have

$$\delta(D,T,T^n x_0) \leq k^n \delta(D,T,x_0). \tag{4.2}$$

By (4.2), we see that for any $m,n \in \mathbf{N}$

$$|D(T^n x_0, T^{n+m}x_0)| \leq \delta(D,T,T^n x_0) \leq k^n \delta(D,T,x_0). \tag{4.3}$$

Since $\delta(D,T,x_0) < \infty$ and $k \in (0,1)$, it follows that

$$\lim_{n\to\infty} |D(T^n x_0, T^{n+m}x_0)| = 0.$$

Hence, $\{T^n x_0\}$ is a Cauchy sequence. Since (X,D) is complete, there exists an element $p \in X$ such that $\{T^n x_0\}$ converges to p.

Suppose that $D(x_0,Tp) \prec \infty$ and $D(p,Tp) \prec \infty$. Then for any $m,n \in \mathbf{N}$

$$|D(T^n x_0, T^{n+m}x_0)| \leq k^n \delta(D,T,x_0). \tag{4.4}$$

From (4.3) and the property (4) in Definition 2.11, there exists $0 \prec \gamma$ such that

$$D(p,T^n x_0) \leq r \limsup_{m\to\infty} |D(T^n x_0, T^{n+m}x_0)| \leq \gamma k^n \delta(D,T,x_0), \tag{4.5}$$

for all $n \in \mathbf{N}$. Consider,

$$D(Tx_0,Tp) \preceq k \max\{D(x_0,p), D(x_0,Tx_0), D(p,Tp), D(Tx_0,p), D(x_0,Tp)\}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we get

$$\begin{aligned} D(Tx_0,Tp) &\preceq k \max\{D(x_0,p), D(x_0,Tx_0), D(p,Tp), rk\delta(D,T,x_0), D(x_0,Tp)\} \\ &= kM_1, \end{aligned} \tag{4.7}$$

where $M_1 = \max\{D(x_0,p), D(x_0,Tx_0), D(p,Tp), rk\delta(D,T,x_0), D(x_0,Tp)\} \prec \infty$. Using the above inequality (4.3), (4.5), (4.7) and Lemma 2.7, we have complex number $0 \prec c_1$ such that

$$\begin{aligned} D(T^2x_0,Tp) &\preceq k \max\left\{D(Tx_0,p), D(Tx_0,T^2x_0), D(p,Tp), D(T^2x_0,p), D(Tx_0,Tp)\right\} \\ &\preceq \max\left\{\gamma k^2\delta(D,T,x_0), c_1k^2|D(Tx_0,T^2x_0)|, kD(p,Tp), \gamma k^2\delta(D,T,x_0), k^2M_1\right\} \\ &\preceq \max\left\{\gamma k^2\delta(D,T,x_0), c_1k^2\delta(D,T,x_0), kD(p,Tp), \gamma k^2\delta(D,T,x_0), k^2M_1\right\} \\ &= \max\{k^2M_2, kD(p,Tp)\}, \end{aligned}$$

where $M_2 = \max\{\gamma\delta(D,T,x_0), c_1\delta(D,T,x_0), M_1\} \prec \infty$. Since $k < 1$, it follows that

$$kD(T^2x_0, Tp) \prec D(T^2x_0, Tp) \preceq \max\{k^2M_2, kD(p,Tp)\}. \tag{4.8}$$

Again, using the above inequality (4.3), (4.5), (4.7), (4.8) and Lemma 2.7, we have complex number $0 \prec c_2$ such that

$$\begin{aligned} D(T^3x_0, Tp) &\preceq k\max\Big\{D(T^2x_0,p), D(T^2x_0,T^3x_0), D(p,Tp), D(T^2x_0,Tp), \\ &\qquad D(p,T^3x_0)\Big\} \\ &\preceq \max\Big\{\gamma k^3\delta(D,T,x_0), c_2k^3|D(T^2x_0,T^3x_0)|, kD(p,Tp), \\ &\qquad kD(T^2x_0,Tp), \gamma k^3\delta(D,T,x_0)\Big\} \\ &\preceq \max\{k^3M_3, kD(p,Tp)\}, \end{aligned}$$

where $M_3 = \max\{\gamma\delta(D,T,x_0), c_2\delta(D,T,x_0), M_2\} \prec \infty$. Continuing this process, by induction above inequality and Lemma 2.7, we have

$$D(T^nx_0, Tp) \preceq \{k^nM, kD(p,Tp)\}, \tag{4.9}$$

for every $n \geq 1$ and $M = \max\{M_1, M_2, ..., M_n\} \prec \infty$. Since $D(x_0, Tp) \prec \infty$ and $D(p,Tp) \prec \infty$, we have

$$\limsup_{n\to\infty} |D(T^nx_0, Tp)| \leq k|D(p,Tp)|. \tag{4.10}$$

By Definition 2.11 (4), there exists $0 \prec r$ such that

$$D(Tp,p) \preceq r\limsup_{n\to\infty} |D(Tp, T^nx_0)|. \tag{4.11}$$

By remark 2.6 (ii) and (4.10), we have

$$|D(Tp,p)| \leq |r|\limsup_{n\to\infty} |D(Tp,T^nx_0)| \leq |r|k|D(p,Tp)|. \tag{4.12}$$

Since $|r|k < 1$, we have $|D(Tp,p)| = 0$ thus $D(Tp,p) = 0$ it follows that p is a fixed point of T, and then

$$D(p,p) = D(Tp,p) = 0. \tag{4.13}$$

If $p^{'}$ is any fixed point of T such that $|D(p,p^{'})| < \infty$ and $|D(p^{'},p^{'})| < \infty$. From Proposition 4.1, we have $D(p^{'},p^{'}) = 0$ and then

$$\begin{aligned} D(p,p^{'}) &= D(Tp,Tp^{'}) \\ &\preceq k\max\Big\{D(p,p^{'}), D(p,Tp), D(p^{'},Tp^{'}), D(Tp,p^{'}), D(p,Tp^{'})\Big\} \\ &\preceq k\max\Big\{D(p,p^{'}), D(p,p), D(p^{'},p^{'}), D(p,p^{'}), D(p,p^{'})\Big\} \\ &\preceq kD(p,p^{'}). \end{aligned}$$

By remark 2.6 (ii), we have

$$|D(p,p^{'})| \leq k|D(p,p^{'})|.$$

Since $k < 1$, we have $|D(p,p^{'})| = 0$ thus $D(p,p^{'}) = 0$. Hence, $p = p^{'}$. This proof is complete.

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