Journal of Nonlinear Analysis and Optimization
Volume 15(1) (2024)
http://ph03.tci-thaijo.org
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

FIXED POINT THEOREM FOR ĆIRIĆ'S QUASICONTRACTION IN GENERALIZED COMPLEX VALUED METRIC SPACES

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ABSTRACT. In this work, we introduce some property of the generalized complex valued metric space and we extend some fixed point results thai is Ćirić's fixed point theorem. Some are recover various complex valued metric space and complex valued b-metric space. Our results extended and improve some results of Mohamed Jleli and Bessem Samet [17].

KEYWORDS: generalized complex valued metric space, Ćirić's quasicontraction.
AMS Subject Classification: 47H10; 54H25; 30L15

1. Introduction

The fixed points theorems has been studied by many mathematicians and fixed Points theorems in metric spaces was introduceed in 1906 [**?**] by Fréchet. In 1922, Banach [6], introduced a fixed point theorem in metric space for contraction mapping.

In recent years, many researcher proved the fixed points theorem in generalizations of metric spaces, see example [9, 3, 20] and references therein. The notion of dislocated metric spaces was introduced in 2000 by Hitzler and Seda [21], see [4]-[18] and references therein.

Very recently, A. Azam, B. Fisher and M. Khan [2] defined the definition of notion of complex valued metric spaces and prove the common fixed point theorems in complex valued metric spaces of a pair of mappings satisfying a contractive condition.

Recently, Jleli and Samet [13], introduce a new concept of generalized metric spaces for which we extend some well-known fixed point results including Banach contraction principle. In 2017, Elkouch and Marhrani [15], proved the existence results for the Kannan contraction in generalized metric space.

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Article history : Received 22 February 2024 Accepted 13 March 2024.

In this paper, motivate by Elkouch and Marhraniwe [15], present a generalized complex valued metric space and prove the relationship between this space with complex valued b-metric space, complex valued dislocated metric space and complex valued metric space. In the final section, we prove the fixed point theorem for a mapping T with satisfying the Ćirić's k-quasicontraction.

2. Preliminaries

In this section, we give some definitions and lemmas for this work.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a **metric space**, and d is called a metric on X.

In 2000, Hitzler and Seda [21], introduce the notion of dislocated metric space as follows.

Definition 2.2. [21] Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a dislocated metric on X if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0 \quad \Rightarrow \quad x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a **dislocated metric space**.

It is easy to show that, the metric space X is dislocated metric space.

Next, we suppose the definition of b-metric space, this space is generalized than metric spaces.

Definition 2.3. [1] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a **b-metric space**. The number $s \geq 1$ is called the coefficient of (X, d).

The following is some example for b-metric spaces.

Example 2.4. [1] Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

In 2017, Elkouch and Marhrani [15] defined a new class of metric space, let X be a nonempty set, and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to \infty} D(x_n, x) = 0 \right\}.$$

Definition 2.5. ([15]) A mapping D is called a generalized metric if it satisfies the following conditions

1. For every $(x, y) \in X \times X$, we have

$$D(x, y) = 0 \Rightarrow x = y.$$

2. For every $(x, y) \in X \times X$, we have

$$D(x, y) = D(y, x).$$

3. There exists a real constant $C > 0$ such that for all $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$, we have

$$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y).$$

The pair (X, D) is called a **generalized metric space**.

It is not difficult to observe that metric d in Definition 2.1 satisfies all the conditions $(i) - (iii)$ with $C = 1$. In 2015 Mohamed Jleli and Bessem Samet [17] prove that any dislocated metric space is is a generalized metric and any b-metric on X is a generalized metric on X.

In this work we will study the generalized metric space in a complex form. Let $\mathbf{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbf{C}$. Define a partial order relation $\preceq$ on $\mathbf{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \preceq z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \preceq z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$ i.e. one of (2),(3) and (4) is satisfied and we will write $z_1 \prec z_2$ only (4) is satisfied.

Remark 2.6. We can easily to check the following:

(i) If $a, b \in \mathbf{R}, 0 \leq a \leq b$ and $z_1 \preceq z_2$ then $az_1 \preceq bz_2, \forall z_1, z_2 \in \mathbf{C}$.
(ii) $0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \preceq z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam et al. [2] defined the complex valued metric space in the following way:

Lemma 2.7. *For any $z \in \mathbf{C}$ with $0 \prec z$ then there exists $r \in \mathbf{C}$ with $0 \prec r$ such that $z = r|z|$.*

Proof Let $z \in \mathbf{C}$ with $0 \prec z$. Put $r = \frac{Re(z)}{|z|} + \frac{Im(z)}{|z|} i \succ 0$. It implied that

$$\begin{aligned} z &= Re(z) + Im(z)i \\ &= \frac{Re(z)}{|z|} \cdot |z| + \frac{Im(z)}{|z|} i \cdot |z| \\ &= \left[\frac{Re(z)}{|z|} + \frac{Im(z)}{|z|} i\right] |z| \\ &= r \cdot |z| \end{aligned}$$

This complete the proof.

[2] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbf{C}$ satisfies the following conditions:

$(C1)$ $0 \preceq d(x, y),$ for all $x, y \in X,$ and $d(x, y) = 0$ if and only if $x = y$;

$(C2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(C3)$ $d(x, y) \preceq d(x, z) + d(z, y),$ for all $x, y \in X$.

Then d is called a complex valued metric on X and (X, d) is called a **complex valued metric space**.

Definition 2.8. Let X be a nonempty set. A function $d : X \times X \to \mathbf{C}$ is called a complex valued dislocated metric on X if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0 \quad \Rightarrow \quad x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \preceq d(x, y) + d(y, z)$.
The pair (X, d) is called a **complex valued dislocated metric space**.

Definition 2.9. [23] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbf{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in \mathbf{C}$, the following conditions are satisfied:
(i) $0 \preceq d(x, y)$
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, z) \preceq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space which is defined in Definition **??**. The following example is some example of complex valued b-metric space.

Example 2.10. [23] Let $X = \mathbf{C}$. Define the mapping $d : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbf{C}, d)$ is complex valued b-metriic space with $s = 2$.

In this work, we consider a nonempty set X, and $D : X \times X \to \mathbf{C}$ be a given mapping. For every $x \in X$, we define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} |D(x_n, x)| = 0 \right\}.$$

Definition 2.11. Let X be a nonempty set, a mapping $D : X \times X \to \mathbf{C}$ is called a generalized complex value metric if it satisfies the following condition
1. For every $x, y \in X$, we have

$$0 \preceq D(x, y).$$

2. For every $x, y \in X$, we have

$$D(x, y) = 0 \Rightarrow x = y.$$

3. For all $x, y \in X$, we have

$$D(x, y) = D(y, x).$$

4. There exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and $\{x_n\} \in C(D, X, x)$, we have

$$D(x, y) \preceq r \limsup_{n\to\infty} |D(x_n, y)|.$$

Then a pair (X, D) is called a **generalized complex valued metric space**.

Definition 2.12. [15] Let (X, D) be a generalized complex valued metric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ is converge to x in X, if $\{x_n\} \in C(D, X, x)$. We denote by $\lim_{n\to\infty} x_n = x$.

Example 2.13. [12] Let $X = [0, 1]$ and let $D : X \times X \to \mathbf{C}$ be the mapping define by for any $x, y \in X$

$$\begin{cases} D(x,y) = (x+y)i; x \neq 0 \text{ and } y \neq 0 \\ D(x,0) = D(0,x) = \frac{x}{2}i \end{cases}$$

Proof Let $x, y \in X$, we have $x \geq 0$ and $y \geq 0$, thus $x + y \geq 0$.
If $D(x, y) = (x + y)i = 0 + (x + y)i \succeq 0 + 0i = 0$.
If $D(x, 0) = \frac{x}{2}i = 0 + \frac{x}{2}i \succeq 0 + 0i = 0$.

$$\text{Hence } D(x,y) \succeq 0.$$

If $D(x, y) = 0$, then $(x + y)i = 0$. Hence, $x = 0 = y$.
If $x \neq 0$ and $y \neq 0$, $D(x, y) = (x+y)i = (y+x)i = D(y, x)$ and $D(x, 0) = D(0, x)$.
Let $\{x_n\} = \{\frac{(n-1)x}{n}\} \subseteq X$, we see that $\limsup_{n\to\infty} |D(x_n, x)| = 0$ and put $r = i$, then we have

$$D(0,y) = \frac{y}{2}i \text{ and } \limsup_{n\to\infty} |D(x_n,y)| = \limsup_{n\to\infty} \sqrt{(\frac{(n-1)x}{n} + y)^2} = x + y.$$

Hence, $D(0, y) = \frac{y}{2}i \preceq (x + y)i$, and we see that

$$D(x,y) = (x+y)i \text{ and } \limsup_{n\to\infty} |D(x_n,y)| = \limsup_{n\to\infty} \sqrt{(\frac{(n-1)x}{n} + y)^2} = x + y.$$

Hence, $D(x, y) = (x + y)i \preceq r \limsup_{n\to\infty} |D(x_n, y)|$.

Definition 2.14. [15] Let (X, D) be a generalized complex valued metric space. Then a sequence $\{x_n\}$ in X is said to Cauchy sequence in X, if $\lim_{n\to\infty} |D(x_n, x_{n+m})| = 0$.

Definition 2.15. [15] Let (X, D) be a generalized complex valued metric space. If every Cauchy sequence is convergent in X then (X, D) is called a complete complex valued metric space.

Definition 2.16. [19] The max function for complex numbers with partial order relation $\preceq$ is defined as
(i) $\max\{z_1, z_2\} = z_2 \Rightarrow z_1 \preceq z_2$;
(ii) $z_1 \preceq \max\{z_1, z_2\} \Rightarrow z_1 \preceq z_2$ or $z_1 \preceq z_3$.
On the similar lines Singh et al. [22] defined min function as
(i) $\min\{z_1, z_2\} = z_1 \Rightarrow z_1 \preceq z_2$;
(ii) $\min\{z_1, z_2\} \preceq z_3 \Rightarrow z_1 \preceq z_3$ or $z_2 \preceq z_3$. Now we introduce the best proximity point and some related concept in complex valued rectangular metric space.

3. Some Property on Generalized Complex Valued Metric Space

In this section we prove some propositions for use in the main theorem and prove some fixed point theorem in generalized complex valued metric space.

Proposition 3.1. *Let (X,D) be a generalized complex valued metric space. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.*

Proof Suppose that $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, by Definition 2.12 we have

$$|D(x_n, x)| \to 0, \ |D(x_n, y)| \to 0, \text{ as } n \to \infty.$$

Using the property (4) in Definition 2.11, we have there exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and since $\{x_n\} \in C(D, X, x)$ such that

$$D(x, y) \preceq r \limsup_{n\to\infty} |D(x_n, y)|.$$

Hence, $D(x, y) = 0$. Using property (2) in Definition 2.11, we have $x = y$.

Proposition 3.2. *Any complex valued b-metric space is a generalized complex valued metric space on X.*

Proof Let $\{x_n\} \in C(d, X, x)$. From the Definition 2.9(iv), we have

$$d(x, y) \preceq s[d(x, x_n) + d(x_n, y)].$$

It follows that, from Lemma 2.7, we have there exists $r_1, r_2 \in \mathbf{C}$ with $0 \prec r_1, r_2$ such that

$$\begin{aligned} d(x, x_n) &= r_1 |d(x, x_n)| \\ d(x_n, y) &= r_2 |d(x_n, y)|. \end{aligned}$$

Then

$$d(x, y) \preceq s[r_1 |d(x, x_n)| + r_2 |d(x_n, y)|].$$

From $\{x_n\} \in C(d, X, x)$, we have

$$d(x, y) \preceq s r_2 \limsup_{n\to\infty} |d(x_n, y)|].$$

Since, $0 \prec r_2$ and $0 \prec s$ then $r = s r_2 \succ 0$ such that

$$d(x, y) \preceq r \limsup_{n\to\infty} |d(x_n, y)|].$$

Hence (X, d) is a generalized complex valued metric space.

It is not difficult to observe that the complex valued metric d satisfies (1-4) of Definition 2.11 and any complex valued dislocated metric space is generalized complex valued metric space.

4. Ćirić's quasicontraction in Generalized Complex Valued Metric Space

In 1974, Ćirić's [11] introduced a class of self-maps on a metric space (X, d) which satisfy the following condition:

$$d(Sx, Sy) \preceq q \max \{d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\}, \tag{4.1}$$

for every $x, y \in X$ and $0 \leq q < 1$. The maps satisfying Condition 4.1 are said to be quasicontractions.

In this section we extend Ćirić's fixed point theorem for quasicontraction is a self-maps on generalized complex valued metric space (X, D) defined by:

$$D(Tx, Ty) \preceq k \max \{D(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\},$$

for every $x, y \in X$ and $k \in (0, 1)$. We say that T is a k-quasicontraction

Proposition 4.1. *Suppose that $T : X \to X$ is a k-quasicontraction for some $k \in (0, 1)$. Then any fixed point $p \in X$ of T satisfies*

$$|D(p, p)| < \infty \Rightarrow D(p, p) = 0.$$

Proof Let $p \in X$ be a fixed point of T such that $|D(p,p)| < \infty$. Since T is a k-quasicontraction for some $k \in (0,1)$, we have

$$\begin{aligned} D(p,p) = D(Tp,Tp) &\preceq k \max\{D(p,p), D(p,Tp), D(p,Tp), D(p,Tp), D(p,Tp)\} \\ &= kD(p,p). \end{aligned}$$

From Remark 2.6(ii), we have

$$|D(p,p)| \leq k|D(p,p)|.$$

Since $k \in (0,1)$, we get $D(p,p) = 0$. This proof is complete.

Next, we suppose that, for every $x \in X$

$$\delta(D,T,x) = \sup\left\{|D(T^i x, T^j x)| : i,j \in \mathbf{N}\right\}.$$

From Proposition 4.1 we have the following result.

Theorem 4.2. *Let (X,D) be a complete generalized complex valued metric space, and let $T : X \to X$ is a k-quasicontraction for some $k \in \left(0, \inf\{1, \frac{1}{|r|}\}\right)$ and there exists element $x_0 \in X$ such that $\delta(D,T,x_0) < \infty$. Then the sequence $\{T^n x_0\}$ converges to some $p \in X$.*

If $D(x_0,Tp) \prec \infty$ and $D(p,Tp) \prec \infty$, then p is a fixed point of T. Moreover, If $p^{'}$ is a fixed point of T in X such that $|D(p,p^{'})| < \infty$ and $|D(p^{'},p^{'})| < \infty$ then $p = p^{'}$.

Proof Let $n \in \mathbf{N}$, for all $i,j \in \mathbf{N}$, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) = D(T(T^{n+i-1}x_0), T(T^{n+j-1}x_0)).$$

By Definition of quasicontraction, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \preceq k \max\left\{\begin{array}{l} D(T^{n+i-1}x_0, T^{n+j-1}x_0), D(T^{n+i-1}x_0, T^{n+i}x_0), \\ D(T^{n+j-1}x_0, T^{n+j}x_0), D(T^{n+j-1}x_0, T^{n+i}x_0), \\ D(T^{n+i-1}x_0, T^{n+j}x_0) \end{array}\right\}$$

Then we have

$$\delta(D,T,T^n x_0) \leq k\delta(D,T,T^{n-1}x_0).$$

Hence, for any $n \geq 1$, we have

$$\delta(D,T,T^n x_0) \leq k^n \delta(D,T,x_0). \tag{4.2}$$

By (4.2) we see that for any $m,n \in \mathbf{N}$

$$|D(T^n x_0, T^{n+m}x_0)| \leq \delta(D,T,T^n x_0) \leq k^n \delta(D,T,x_0). \tag{4.3}$$

Since $\delta(D,T,x_0) < \infty$ and $k \in (0,1)$, it follows that

$$\lim_{n\to\infty} |D(T^n x_0, T^{n+m}x_0)| = 0.$$

Hence $\{T^n x_0\}$ is a Cauchy sequence. Since (X,D) is complete, there exists a element $p \in X$ such that $\{T^n x_0\}$ convergent to p.

Suppose that $D(x_0,Tp) \prec \infty$ and $D(p,Tp) \prec \infty$. Then for any $m,n \in \mathbf{N}$

$$|D(T^n x_0, T^{n+m}x_0)| \leq k^n \delta(D,T,x_0). \tag{4.4}$$

From (4.3) and the property (4) in Definition 2.11, there exists $0 \prec \gamma$ such that

$$D(p,T^n x_0) \leq r \limsup_{m\to\infty} |D(T^n x_0, T^{n+m}x_0)| \leq \gamma k^n \delta(D,T,x_0), \tag{4.5}$$

for all $n \in \mathbf{N}$. Consider,

$$D(Tx_0,Tp) \preceq k \max\{D(x_0,p), D(x_0,Tx_0), D(p,Tp), D(Tx_0,p), D(x_0,Tp)\}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we get

$$\begin{aligned} D(Tx_0, Tp) &\preceq k \max\{D(x_0, p), D(x_0, Tx_0), D(p, Tp), rk\delta(D, T, x_0), D(x_0, Tp)\} \\ &= kM_1, \end{aligned} \tag{4.7}$$

where $M_1 = \max\{D(x_0, p), D(x_0, Tx_0), D(p, Tp), rk\delta(D, T, x_0), D(x_0, Tp)\} \prec \infty$. Using the above inequality (4.3), (4.5), (4.7) and Lemma 2.7, we have complex number $0 \prec c_1$ such that

$$\begin{aligned} D(T^2x_0, Tp) &\preceq k \max\{D(Tx_0, p), D(Tx_0, T^2x_0), D(p, Tp), D(T^2x_0, p), D(Tx_0, Tp)\} \\ &\preceq \max\{\gamma k^2\delta(D, T, x_0), c_1k^2|D(Tx_0, T^2x_0)|, kD(p, Tp), \gamma k^2\delta(D, T, x_0), k^2M_1\} \\ &\preceq \max\{\gamma k^2\delta(D, T, x_0), c_1k^2\delta(D, T, x_0), kD(p, Tp), \gamma k^2\delta(D, T, x_0), k^2M_1\} \\ &= \max\{k^2M_2, kD(p, Tp)\}, \end{aligned}$$

where $M_2 = \max\{\gamma\delta(D, T, x_0), c_1\delta(D, T, x_0), M_1\} \prec \infty$. Since, $k < 1$ it follows that

$$kD(T^2x_0, Tp) \prec D(T^2x_0, Tp) \preceq \max\{k^2M_2, kD(p, Tp)\}. \tag{4.8}$$

Again, using the above inequality (4.3), (4.5), (4.7), (4.8) and Lemma 2.7, we have complex number $0 \prec c_2$ such that

$$\begin{aligned} D(T^3x_0, Tp) &\preceq k \max\Big\{D(T^2x_0, p), D(T^2x_0, T^3x_0), D(p, Tp), D(T^2x_0, Tp), \\ &\qquad D(p, T^3x_0)\Big\} \\ &\preceq \max\Big\{\gamma k^3\delta(D, T, x_0), c_2k^3|D(T^2x_0, T^3x_0)|, kD(p, Tp), \\ &\qquad kD(T^2x_0, Tp), \gamma k^3\delta(D, T, x_0)\Big\} \\ &\preceq \max\{k^3M_3, kD(p, Tp)\}, \end{aligned}$$

where $M_3 = \max\{\gamma\delta(D, T, x_0), c_2\delta(D, T, x_0), M_2\} \prec \infty$. Continuing this process, by induction above inequality and Lemma 2.7, we have

$$D(T^nx_0, Tp) \preceq \{k^nM, kD(p, Tp)\}, \tag{4.9}$$

for every $n \geq 1$ and $M = \max\{M_1, M_2, ..., M_n\} \prec \infty$. Since $D(x_0, Tp) \prec \infty$ and $D(p, Tp) \prec \infty$, we have

$$\limsup_{n\to\infty} |D(T^nx_0, Tp)| \leq k|D(p, Tp)|. \tag{4.10}$$

By Definition 2.11 (4), there exists $0 \prec r$ such that

$$D(Tp, p) \preceq r \limsup_{n\to\infty} |D(Tp, T^nx_0)|. \tag{4.11}$$

By remark 2.6 (ii) and (4.10), we have

$$|D(Tp, p)| \leq |r| \limsup_{n\to\infty} |D(Tp, T^nx_0)| \leq |r|k|D(p, Tp)|. \tag{4.12}$$

Since $|r|k < 1$, we have $|D(Tp, p)| = 0$ thus $D(Tp, p) = 0$ it follows that p is a fixed point of T, and then

$$D(p, p) = D(Tp, p) = 0. \tag{4.13}$$

If p' is any fixed point of T such that $|D(p, p')| < \infty$ and $|D(p', p')| < \infty$. From Proposition 4.1 we have $D(p', p') = 0$ and then

$$\begin{aligned} D(p, p') &= D(Tp, Tp') \\ &\preceq k \max\Big\{D(p, p'), D(p, Tp), D(p', Tp'), D(Tp, p'), D(p, Tp')\Big\} \end{aligned}$$

$$\preceq \quad k \max\left\{D(p,p^{'}), D(p,p), D(p^{'},p^{'}), D(p,p^{'}), D(p,p^{'})\right\}$$
$$\preceq \quad kD(p,p^{'}).$$

By remark 2.6 (ii), we have

$$|D(p,p^{'})| \leq k|D(p,p^{'})|.$$

Since $k < 1$, we have $|D(p,p^{'})| = 0$ thus $D(p,p^{'}) = 0$. Hence $p = p^{'}$. This proof is complete.

5. Acknowledgements

I would like to thank Thailand Science Research and Innovation for financial support.

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