S. Park, J. Nonlinear Anal. Optim. Vol. 15(1) (2024), 1-11

Journal of Nonlinear Analysis and Optimization
Volume 15(1) (2024)
http://ph03.tci-thaijo.org
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

EPISODES IN METRIC FIXED POINT THEORY RELATED TO F. E. BROWDER

SEHIE PARK*[1]

[1] The National Academy of Sciences, Republic of Korea, Seoul 06579;
Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea

---

ABSTRACT. F.E. Browder in 1979 posed a fixed point theorem of great generality and complexity such that a large part of the literature on contractive type of maps can be subsumed under an intuitive and simple mode of argument. Immediately after, several researchers found better theorems than his result. Since then, only a few authors have quoted his theorem. However, until recently, certain minor papers related to Browder's aim. Our aim in this paper is to introduce the contents of them.

KEYWORDS: fixed point, complete metric space, contractive type conditions
AMS Subject Classification: 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

---

1. Prologue

In 1979, Felix Browder [4] began as follows: "The contraction mapping principle (known under a number of names, including those of Picard, Banach, and Caccioppoli) has long been one of the simplest and most useful tools in the study of nonlinear problems. During the past two decades, the development of nonlinear functional analysis has taken a diversity of forms and used a great variety of more sophisticated tools and methods. Yet, in many of the theories that have been created, from one point of view, these more sophisticated results can be regarded as far-reaching extensions of the contraction mapping principle in contexts of richer structure.

Over the same decades, however, there has grown an extensive literature devoted to sharper forms of the contraction mapping principle on its native terrain, i.e. for mappings of complete metric spaces. As can be seen from a recent survey by Rhoades [17] (in which 149 different conditions are analyzed and compared), this literature has reached a point of such scholastic complexity and unreadability that its usefulness is open to serious question."

---

* *Corresponding author.*
Email address : park35@snu.ac.kr; sehiepark@gmail.com.
Article history : Received 5 February 2024; Accepted 5 May 2024.

Forty-five years later, in 2024, the situation became more serious than in Browder's time. In fact, there have been appeared hundreds of contractive type conditions and thousands of new artificial metric type spaces. Note that the present writers have never made any new contractive conditions nor any new artificial spaces in the metric fixed point theory.

Browder [4] made a theorem in order to unify known fixed point theorems until that time. Immediately after [4] appeared, Hegedŝ and Szilágyi [7] published a very general metric fixed point theorem. Based on their work, the present writer [14], in 1980, classified contractive type conditions in order to improve the survey by Billy E. Rhoades [17]. Without recognizing [7] and [14], Walter [20] improved Browder's result. However, Park [15] did the same work based on [7] and [14]. Since then, there have appeared several papers closely related to Browder's.

In this paper, we recall the contents of such articles related to the lost paper of Browder as small episodes in the history of the metric fixed point theory. All the quoted papers in this paper are introduced in chronological order.

Now we begin with the following:

2. Browder [4] in 1979

"It is our purpose in the present discussion to show that a large part of this complexity can be subsumed under an intuitive and simple mode of argument. We present a fixed point theorem of all great generality and complexity, which includes all the detailed cases that the author has seen in the literature. We deduce this principle in an easily comprehensible way from a simpler more conceptual principle. The proof of the latter is a slight variant of an earlier proof given for contractive fixed point theorems by the writer" in 1968 and 1976.

Browder gives the following definition:

Definition 1. Let Φ be a function from $\mathbb{R}^+$, the non-negative reals, to $\mathbb{R}^+$. Then Φ is said to be a contractive gauge function if $\Phi(0) = 0$, Φ is non-decreasing from the right (i.e. r_j decreasing and converging to $r > 0$ implies that $\Phi(r_j)$ converges to $\Phi(r)$), and

$$\Phi(r) < r$$

for all $r > 0$.

The following is the main result of Browder:

Theorem 2. *Let M be a complete metric space with metric d, M_0 a subset of M. Let f be a continuous mapping of M into M which carries M_0 into M_0. Let Φ be a contractive gauge function in the sense of Definition 1. For each x in M_0, suppose that there exists a positive integer $n(x)$ and for each y in M_0, three subsets $J_1 = J_1(x,y,n), J_2 = J_2(x,y,n), J_3 = J_3(x,y,n)$ of $\mathbb{N}\times\mathbb{N}$ such that for each $n \geq n(x)$, $y \in M_0$,*

$$d(f^n x, f^n y) \leq \Phi(\max(\sup_{[j,k]\in J_1} d(f^j x, f^k y), \sup_{[r,s]\in J_2} d(f^r x, f^s y), \sup_{[m,t]\in J_3} d(f^m x, f^t y))).$$

Then: f has a fixed point x_0 in M such that for each x in M_0, $f^j x$ converges to x_0 in M as $j \longrightarrow +\infty$.

In order to prove this, Browder needed the following Propositions:

Proposition 2. *Let M be a complete metric space, M_0 a subset of M, f a continuous mapping of M into M which carries M_0 into M_0. Let Φ be a contractive gauge function. Let $O(f,x)$ denote the orbit of x under f, i.e. $O(f,x) = \bigcup_{j\geq 0}\{f^j x\}$.*

Suppose that for each x in M_0, there exists $n(x)$ a positive integer such that $\operatorname{diam} O(f, f^{n(x)}x)) \leq \Phi(\operatorname{diam} O(f,x)))$.

Then f has a fixed point in $\bigcap_{j\geq 0} \operatorname{cl}(O(f, f^j x))$ for each x in M_0.

Proposition 3. *Let M be a complete metric space, M_0 a subset of M, f a continuous mapping of M into M which carries M_0 into M_0. Let Φ be a contractive gauge function as in Definition 1. For any pair $\{x, y\}$ in M, let $O(f, x, y)$ be the orbit of the pair under f, i.e.*

$$O(f, x, y) = \bigcup_{j\geq 0} \{f^j x\} \cup \{f^j y\}.$$

Suppose that for each x in M_0, there exists a positive integer $n(x)$ such that for any pair x, y in M_0,

$$\operatorname{diam}\left(O(f, f^{n(x)+n(y)}(x), f^{n(x)+n(y)}(y))\right) \leq \Phi(\operatorname{diam} O(f, x, y))).$$

Then for each x in M_0, $f^j x$ converges to a fixed point of f in M, and this fixed point is independent of the choice of x in M_0.

In Erratum to this paper in 1981, Browder [5] stated: "In my paper [4], the results as stated are not valid without the additional assumption that all the orbits are bounded (as an assumption which is applied implicitly throughout the proofs). With this additional assumption, the corrected theorems are indeed valid. However, in terms of the original intention of the paper to include all the principal results in the catalogue of contractive fixed point theorems of Rhoades [17], this requires stronger hypotheses to ensure the orbit boundedness from contractivity-type hypotheses. A detailed treatment of such results (as well as of the possibility of dropping continuity hypotheses) is given in the paper by W. Walter [20]."

However, only a few quoted his theorem.

3. Hegedüs and Szilágyi [7] in 1980

Independently of Browder's work, the following appeared:

Theorem HS. *Let X be a complete metric space, and f a selfmap of X such that* $\operatorname{diam}(O(x)) < \infty$ *for any $x \in X$. Suppose that for any x, $y \in X$,*

(HS) *for any $\varepsilon > 0$, there exist a ε_0 and a $\delta > 0$ such that $0 < \varepsilon_0 < \varepsilon$ and*

$$\varepsilon \leq \operatorname{diam}(O(x) < \varepsilon + \delta \quad \text{implies} \quad d(fx, fy) \leq \varepsilon_0.$$

Then f has a unique fixed point $p \in X$ and $f^n x$ converges to p for each $x \in X$.

For a non-continuous selfmap, this theorem is one of the sharpest results.

4. Kasahara [10] in 1980

Kasahara obtained the following (formulated by Park [15]):

Proposition 2. *Let X be a complete metric space, Y a subset of X, f a continuous selfmap of X such that $fY \subset Y$ and* $\operatorname{diam}(O(x)) < \infty$ *for any $x \in Y$. Suppose that for each $x \in Y$, there exists a positive integer $\sharp x$ such that*

(v) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \operatorname{diam}(O(x)) < \varepsilon + \delta \text{ implies } \operatorname{diam}(O(f^{\sharp x}x)) < \varepsilon.$$

Then f has a fixed point in $\bigcap_{n\geq 0} \overline{O(f^n x)}$ for each $x \in Y$.

Proposition 3. *Let X be a complete metric space, Y a subset of X, f a continuous selfmap of X such that $fY \subset Y$ and* $\operatorname{diam}(O(x)) < \infty$ *for any $x \in Y$. Suppose that* *for each $x \in Y$, there exists a positive integer $\sharp x$ such that for any pair $x,\ y \in Y$, the following condition holds:*

(vi) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \operatorname{diam}(O(x,y)) < \varepsilon + \delta \text{ implies } \operatorname{diam}(O(f^{\sharp x+\sharp y}x, f^{\sharp x+\sharp y}y)) < \varepsilon.$$

Then for each $x \in Y$, $\{f^n x\}$ converges to a fixed point of f in X, and this fixed point is independent of the choice of $x \in Y$.

Influenced by Hegedüs and Szilágyi [7], Kasahara noticed that his propositions extend the corresponding ones of Browder [4]. Note also that Browder had to assume that each orbit in Y is bounded in his Propositions 2 and 3. In Erratum [5] to his paper, Browder in 1981 agreed this fact.

5. Park [14] in 1980

After the comparative study of Billy Rhoades on contractive conditions [17], there have appeared wider classes of mappings of the form

$$d(fx, fy) < \operatorname{diam}(O(x) \cup O(y)),$$

where f is a selfmap of a metric space (X, d). A point $x \in X$ is said to be regular for f if $\operatorname{diam} O(x) < \infty$.

Given $x,\ y \in X$, let

$$m(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\} \text{ and}$$

$$\delta(x,y) = \operatorname{diam}\{O(x) \cup O(y)\}.$$

In order to update Rhoades' program, we made a classification of contractive type conditions in [14]. The following is part of them:

(C) Given $\varepsilon > 0$, there exist $\epsilon_0 < \epsilon$ and $\delta_0 > 0$ such that for any $x,\ y \in X$,

(Cd) $\varepsilon \leq d(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$.
(Cm) $\varepsilon \leq m(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$.
(Cδ) $\varepsilon \leq \delta(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$. (Hegedüs-Szilágyi [7]).

(D) There exists a nondecreasing right continuous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that $\phi(t) < t$ for $t > 0$ and, for any $x,\ y \in X$,

(Dd) $d(fx,fy) \leq \phi(d(x,y))$.
(Dm) $d(fx,fy) \leq \phi(m(x,y))$.
(Dδ) $d(fx,fy) \leq \phi(\delta(x,y))$ if $x,\ y$ are regular. (Hegedüs-Szilágyi)

Now we give a theorem for a map satisfying condition (Cδ).

Theorem 2(Cδ). *Let f be a selfmap of a metric space X. Suppose there exists a regular point $u \in X$ such that*
(1) *$O(u)$ has a regular cluster point $p \in X$, and*
(2) *the condition* (Cδ) *holds on $O(u) \cup O(p)$.*

Then f has a unique fixed point p in $\overline{O(u)}$ and $f^n u \longrightarrow p$.

In [14], a large number of consequences of Theorem 2(Cδ) were given.

6. WALTER [20] IN 1981

"THROUGHOUT this note, X denotes a complete metric space with distance function d, and T is a map from X into itself. Powers of T are defined by $T^0x = x$ and $T^{n+l}x = T(T^nx),\ n \geq 0$. Occasionally, we use the notation $x^k = T^kx$, in particular $x^0 = x,\ x^1 = Tx$, for the sake of brevity. $\cdots$ The letter ϕ denotes a contractive gauge function, i.e. a continuous, increasing function from $\mathbb{R}_+$, the non-negative reals, into $\mathbb{R}_+$ which satisfies $\phi(s) < s$ for $s > 0$."

" We consider two conclusions:

(FA) T has one and only one fixed point $z \in X$.

(SA) The successive approximations converge, i.e. there exists $z \in X$ such that $d(T^kx, z) \longrightarrow 0$ as $k \longrightarrow \infty$ for any $x \in X$.

Let us have

(C4) $\qquad d(Tx, Ty) \leq \phi(\operatorname{diam} O(x, y))$ for $x,\ y \in X$.

Theorem 2. (C4) implies (FP, SA), if all orbits are bounded."

Instead of the above, we borrow the following from Kirk-Saliga [11]:

"We state Walter's result below. (The underlying ideas are those of Browder [4].) In this theorem, ϕ denotes a contractive gauge function on a metric space M. This means $\phi : R^+ \longrightarrow R^+$ is continuous, nondecreasing, and satisfies $\phi(s) < s$ for $s > 0$. "

Theorem 4.2. *Let M be a complete metric space and suppose $T : M \longrightarrow M$ has bounded orbits and satisfies the following condition. For each $x,\ y \in M$,*

$$d(T(x), T(y)) \leq \phi(\operatorname{diam}(O(x, y))).$$

Then T has a unique fixed point $z \in M$ and $\lim_{k\longrightarrow\infty} T^k(x) = z$ for each $x \in M$.

COMMENTS: Walter showed that Browder's sophisticated contractive type condition can be replaced by rather concise forms. However, Walter's contractive gauge functions are still partial to those in our next paper. In fact, in terms of Park [14] in 1980, Walter adopted the contractive type condition more strictly than the type (D), and Park used the conditions of the type (C) more generally than (D). For the comparison of those conditions, see [10] and [14].

7. PARK [15] IN 1981

Abstract: F.E. Browder [4] posed a fixed point theorem of great generality and complexity such that a large part of the literature on contractive type of maps can be subsumed under an intuitive and simple mode of argument. In this paper, we present sharper forms of such fixed point theorems, which show that Browder's result can be stated in a more general setting and include much more detailed cases than his.

Theorem 1. *Let X be a complete metric space, Y a subset of X, and f a selfmap of X such that $fY \subset Y$ and $\operatorname{diam}(O(x)) < \infty$ for all $x \in Y$. Suppose that for each $x \in Y$ there exists a positive integer $\sharp x$ such that the following condition holds:*

(vii) *for any $\varepsilon > 0$ there exist a $\delta > 0$ and an ε_0 with $0 < \varepsilon_0 < \varepsilon$ such that*

$$x,\ y \in Y,\ \varepsilon \leq \operatorname{diam}(O(x, y)) < \varepsilon + \delta \text{ imply } d(f^mx, f^ny) \leq \varepsilon_0$$

for each $m \geq \sharp x, n \geq \sharp y$.

If f is continuous, then f has a fixed point p in X such that for each $x \in Y$, $\{f^n x\}$ converges to p in X as $n \longrightarrow \infty$.

Theorem 1 is a direct consequences of Proposition 3 in [10], [15] since (vii) implies (vi) there, and extends the main result of [4].

A number of variations of Theorems 1 and HS are possible; see [9].

For non-complete metric spaces Theorems 1 and HS are easily extended as follows:

Theorem 2. *Let f be a selfmap of a metric space X. Suppose that there exists a point $u \in X$ with $\mathrm{diam} O(u) < \infty$ such that*

(1) *$O(u)$ has a cluster point $p \in X$ with $\mathrm{diam} O(p) < \infty$,*
(2) *f is (orbitally) continuous at p, and*
(3) *$Y = O(u, p)$ satisfies the conditions in Theorem 1.*

Then f has a unique fixed point p in $\overline{O(u)}$ and $f^n u \longrightarrow p$ as $n \longrightarrow \infty$.

Then Theorem 2(Cδ) in Park [14] is listed as Theorem 4.

8. Kirk and Saliga [11] in 2000

"Motivated by Browder's elegant unification of numerous diverse contractive conditions, Walter [20] proved a far-reaching extension of Banach's theorem. We use this fact to show that Theorem 4.1 extends to a much wider class of mappings under the additional assumption that the orbits of T are bounded.

Using this fact, we obtain the following. (It seems that the condition initially appeared in a paper of Hegedüs [6].)"

Theorem 4.3. *Let M be a complete metric space and suppose $T : M \longrightarrow M$ has bounded orbits and satisfies: there exists $\alpha < 1$ such that for each $x,\ y \in M$,*

$$d(T(x), T(y)) \leq \alpha\, \mathrm{diam}(O(x, y)) \ for\ all\ x,\ y \in M.$$

Suppose $\{x_n\} \subset M$ satisfies $\lim_n d(x_n, T(x_n)) = 0$. Then T has a unique fixed point $z \in M$, and $\lim_{c \longrightarrow 0^+} \mathrm{diam}(L_c) = 0$. Moreover, $\lim_n d(x_n, T(x_n)) = 0$ if and only if $\lim_n x_n = z$.

Here, $L_c := \{x \in M : F(x) \leq c\}$ for all $c \geq 0$, where $F(x) := d(x, Tx)$ for every $x \in M$. It was noticed in [3] that F is an r.g.i. on M. We recall (see [2]) that a function $G : M \longrightarrow R$ is said to be a regular-global-inf (r.g.i.) at $x \in M$ if $G(x) > \inf_M(G)$ implies there exist $\varepsilon > 0$ such that $\varepsilon < G(x) - \inf_M(G)$ and a neighborhood N_x of x such that $G(y) > G(x) - \varepsilon$ for each $y \in N_x$. If this condition holds for each $x \in M$, then G is said to be an r.g.i. on M:
To prove Theorem 4.3, the authors have used the preceding result of Walter [20].

In the final Section: By taking $y = T(x)$, one has

$$d(T(x), T^2(x)) \leq \alpha\, \mathrm{diam}(O(x, Tx))) = \alpha\, \mathrm{diam}(O(x)) \ for\ all\ x \in M.$$

and this quickly leads to

$$\mathrm{diam}(O(Tx))) = \alpha\, \mathrm{diam}(O(x)) \ for\ all\ x \in M.$$

This can be rewritten as

$$\mathrm{diam}(O(x)) \leq (1 - \alpha)^{-1}[\mathrm{diam}(O(x)) - \mathrm{diam}(O(Tx)))] \ for\ all\ x \in M.$$

Since $d(x, T(x)) \leq \mathrm{diam}(O(x))$, if the mapping $\varphi : M \longrightarrow R$ defined by setting $\varphi(x) = \mathrm{diam}(O(x))$ is lower semicontinuous, then this condition, much weaker than the one in Theorem 4.3, assures that T has a fixed point by the Caristi fixed point theorem.

9. Akkouchi [1] in 2001

Abstract: We prove that the conclusion of a result of Kirk and Saliga [Theorem 4.3] remains valid for a wide class of contractive gauge functions.

2.1. Let Φ be the set of continuous functions $\phi : R^+ \longrightarrow R^+$ such that ϕ is nondecreasing on R^+ and such that the mapping $x \mapsto x - \phi(x)$ from $[0, +\infty[$ onto $[0, +\infty[$ is strictly increasing. We notice that each element ϕ in Φ is a gauge function and that Φ_1 is strictly contained in Φ. Indeed, we can give examples of elements in $\Phi \setminus \Phi_1$.

The following theorem is the main result of this short communication.

Theorem C. *Let (M, d) be a complete metric space and suppose $T : M \longrightarrow M$ has bounded orbits and satisfies the following condition:*

$$d(Tx, Ty) \leq \phi(\operatorname{diam}(O(x, y))) \ for \ all \ x, y \in M,$$

where $\phi \in \Phi$. Then

(1) *T has a unique fixed point $z \in M$, and $\lim_{k \longrightarrow +\infty} T^k(x) = z$ for each $x \in M$.*
(2) $\lim_{c \longrightarrow 0^+} \operatorname{diam}(L_c) = 0$.
(3) *For each sequence $\{x_n\} \subset M$, $\lim_n d(x_n, Tx_n) = 0$ if and only if $\lim_n x_n = z$.*
(4) *The map $F : x \mapsto d(x, Tx)$ is an r.g.i. on M.*

Comments: According to our classification [15] in 1980 or Section 5, the contractive condition belongs to (Dδ). Hence it may be extended to (Cδ).

10. Jachymski [8] in 2001

In 1927, Knaster proved a fixed point theorem for increasing — under set-inclusion — mappings, on and to the family of all subsets of a set. In 1939, Tarski extended Knaster's result to increasing mappings on a complete lattice and he gave its applications in set theory and topology, but his result was unpublished until 1955.

"In the sequel, we shall show how the Knaster-Tarski fixed point theorem can be used to derive some results from metric fixed point theory. We start with Amann's [A] proof (see also Zeidler [Z, p. 512]) of a fixed point theorem for the so-called diametric contractions. In fact, we shall extend his argument by considering a more general class of mappings: A selfmap f of a bounded metric space is said to be a *diametric φ-contraction* if there is a non-decreasing function $\varphi : R_+ \longrightarrow R_+$($R_+$ denotes the set of all nonnegative reals) such that $\lim_{n \longrightarrow \infty} \varphi^n(t) = 0$ for all $t > 0$ (see Matkowski [M] or Dugundji and Granas [DG, p. 12]) and

$$\operatorname{diam} f(A) \leq \varphi(\operatorname{diam} A)$$

for all nonempty, closed and f-invariant subsets A of X.

A more realistic special case here is the following *Walter's contraction* [20], i.e., a mapping f which satisfies the inequality

$$d(fx, fy) \leq \varphi(\operatorname{diam}(O_f(x) \cup O_f(y))) \text{ for all } x, \ y \in X,$$

where $O_f(x) := \{f^{n-1}x : n \in N\}$ is an *orbit* of f at a point x and the function φ is nondecreasing, right continuous and $\varphi(t) < t$ for $t > 0$ (then $\lim_{n \longrightarrow \infty} \varphi^n(t) = 0$; cf. Browder [4]). That such an f satisfies (2.1), follows immediately from the fact that $x \in A$ implies $O_f(x) \subset A$ if A is f-invariant.

Theorem 2.3 *Let (X, d) be a complete bounded metric space and f be a diametric φ-contraction on X. Then f has a unique fixed point.*

Comments: For the references [A, Z, M, DG], see [8]. Jachymski gave a very lengthy proof and some related results. He showed that the Knaster-Tarski fixed point theorem also yields some extensions of the Contraction Principle for mappings on uniform spaces, given by K.-K. Tan, Taraftar, D.H. Tan and Angelov.

11. Proinov [16] in 2006

Proinov unified the Boyd-Wong [BW], Jachymski [J], Matkowski [M] and Meir-Keeler [MK] type contractions (for such references, see [16]) and proved the following interesting generalization of the Banach contraction principle:

Theorem. [16] *Let (E, ρ) be a complete metric space and $f : E \longrightarrow E$ a continuous and asymptotically regular mapping such that*

(1) $\rho(f(x), f(y)) \leq \psi(L(x, y))$ *for all* $x,\ y \in E$,
(2) $\rho(f(x), f(y)) < L(x, y)$ *for all* $x,\ y \in E$ *whenever* $L(x, y) \neq 0$;
where $\mu \geq 0,\ L(x, y) = \rho(x, y) + \mu[\rho(x, f(x)) + \rho(y, f(y))]$, *and* $\psi : [0, \infty) \longrightarrow [0, \infty)$ *is such that for each* $\xi > 0$ *there exists a* $\delta > \xi$ *such that* $\xi < t < \delta$ *implies* $\psi(t) \leq \xi$.

Then f admits a unique fixed point in E. Moreover, if $\mu = 1$ and ψ is continuous with $\psi(t) < t$ for all $t > 0$ then continuity of f is not needed.

Comments: Proinov's theorem seems to be very close to the works of Hegedš-Szilágyi [7] with our conditions (Cδ) and (Dδ) in Section 5.

12. Jachymski [9] in 2009

Abstract: We revisit a fixed point theorem for contractions established by Felix Browder in 1968. We show that many definitions of contractive mappings that appeared in the literature after 1968 turn out to be equivalent formulations or even particular cases of Browder's definition. We also discuss the problem of the existence of approximate fixed points of continuous mappings; in particular, we settle it in the affirmative for Browder contractions. Finally, we recall three problems concerning Browder contractions that remain unsolved.

In Section 2, the author is interested in the existence of approximate fixed points.

Some further extensions of the Banach Principle and their connections with Browder's theorem are discussed in Section 3. A novelty here is Theorem 13 by means of which a recent result of Branciari for mappings satisfying a contractive condition of integral type can be subsumed under Browder's theorem.

The author closes the paper with three questions concerning Browder contractions. They deal with a set-valued version of Browder's theorem (Question 1), the stability of successive approximations (Question 2), and the continuous dependence of fixed points on parameters (Question 3).

13. Tasković [T] in 2009

Abstract: We prove that a result of Kirk and Saliga [11, Theorem 4.2., p.149] has been for the first time proved before 25 years in Tasković [18, Theorem 1, p.250]. But the authors neglected and ignored this historical fact.

From Introduction: In recent years, a great number of papers have presented generalizations of the well-known Banach-Picard contraction principle. Recently, Kirk and Saliga have proved the following statement (see [11, Theorem 4.2., p.149]).

Theorem 1. (Kirk-Saliga [11], Walter [20]) *Let (X,ρ) be a complete metric space and suppose $T : X \longrightarrow X$ has bounded orbits and satisfies the following condition:*

$$\rho[Tx,Ty] \leq \Phi(\mathrm{diam}\{x,y,Tx,Ty,T^2x,T^2y,\dots\})$$

for all $x,\ y \in X$, where $\Phi : \mathbb{R}^0_+ \longrightarrow \mathbb{R}^0_+ := [0,+\infty)$ is a continuous non-decreasing function and satisfies $\Phi(t) < t$ for every $t > 0$. Then T has a unique fixed point $\zeta \in X$ and $\{T^n(a)\}_{n\in\mathbb{N}}$ converges to ζ for every $a \in X$.

In connection with this, in 1980, I proved the following result for a fixed point on metric spaces that has the longest of all known sufficient conditions (linear and nonlinear) for the existing unique fixed point:

Theorem 2. (Tasković [18]) *Let T be a mapping of a metric space (X,ρ) into itself and let X be T-orbitally complete. Suppose that there exists a function $\varphi : \mathbb{R}^0_+ \longrightarrow \mathbb{R}^0_+ := [0,+\infty)$ satisfying*

$$(\forall t \in \mathbb{R}_+ := (0,+\infty))(\varphi(t) < t \text{ and } \lim sup_{z\longrightarrow t+0}\varphi(z) < t)$$

such that

$$\rho[Tx,Ty] \leq \varphi(\mathrm{diam}\{x,y,Tx,Ty,T^2x,T^2y,\dots\}$$

and $\mathrm{diam}\, O(x) \in \mathbb{R}^0_+$ *for all $x,\ y \in X$.*

Then T has a unique fixed point $\zeta \in X$ and $\{T^n(a)\}_{n\in\mathbb{N}}$ converges to ζ for every $a \in X$.

We notice that Theorem 1 is a very special case of Theorem 2.

14. Kumam, Dung, and Sitytithakerngkiet [12] in 2015

Abstract: We state and prove a generalization of Ćirić fixed point theorems in metric space by using a new generalized quasi-contractive map. These theorems extend other well-known fundamental metrical fixed point theorems in the literature (Banach, Kannan, Nadler, Reich, etc.) Moreover, a multi-valued version for generalized quasi-contraction is also established.

Definition 2.4. Let $T : X \longrightarrow X$ be a mapping on metric space X. The mapping T is said to be a *generalized quasi-contraction* iff there exists $q \in [0,1)$ such that for all $x,\ y \in X$,

$$d(Tx,Ty) \leq q\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx),$$
$$d(x,T^2x), d(Tx,T^2x), d(x,T^2y), d(T^2x,Ty)$$

Some corollaries and multi-valued versions of them are added.

Comments: The contractive condition implies the Hegedüs condition $d(Tx,Ty) \leq q\,\mathrm{diam}\,\{O_T(x) \cup O_T(y)\}$. Hence, the main theorem was already known.

15. Pant and Khantwal [13] in 2023

"We present some new existence results for single and multivalued mappings in metric spaces on very general settings. Some illustrative examples are presented to validate our theorems."

Theorem 2.2. *Suppose (E,ρ) is a metric space. Let $f : E \longrightarrow E$ a mapping such that for some $v_0 \in E$,*

$$\frac{1}{2}\rho(x,f(x)) \leq \rho(x,y) \Longrightarrow \rho(f(x),f(y)) \leq \psi(N(x,y))$$

for all $x,\ y \in \overline{O(v_0, f)}$ *with* $x \neq y$*, where*

$$N(x,y) = \max\{\rho(x,y), \rho(x,f(x)), \rho(y,f(y)), \frac{1}{2}[\rho(y,f(x)) + \rho(x,f(y))]\}.$$

If E *is* f*-orbitally complete then the sequence of iterations* $(f^n(v_0))$ *is Cauchy in* E *and converges to the unique fixed point of* f *in* $\overline{O(v_0, f)}$*.*

The authors added six corollaries of the same nature.

16. Epilogue

All the stories in this paper begins with Browder, but no one quotes his theorem (except Jachymski [9] and the present one) and no one practically uses his theorem. However, certain influences from his work still remain from time to time.

In the below, $\Longrightarrow$ means certain influence. Then we have

Browder $\Longrightarrow$ Walter $\Longrightarrow$ Kirk-Saliga $\Longrightarrow$ Akkouchi.

Hegedüs-Szylági $\Longrightarrow$ Kasahara $\Longrightarrow$ Park.

Walter $\Longrightarrow$ Jachymski.

Browder $\Longrightarrow$ Jachymski.

The first and third rows seem to be obsolete.

Especially, Kirk-Saliga mentioned Hegedüs, but not followed him. Art Kirk was a long-time friend of the writer. Jachymski [8] did not mention Hegedüs-Szylági's work. Kasahara made a friendship with the writer just before he passed away. The writer met Browder a long time ago at the conferences at Berkeley and Marseilles.

Almost all the works mentioned in this paper are obsolete. Nowadays, researchers are adopting new terminology, like as weaker spaces than metric ones, orbital completeness, orbital continuity of maps, etc.

References

1. M. Akkouchi, *On a result of W. A. Kirk and L. M. Saliga*, J. Comput. Appl. Math. **142** (2002), 445–448.
2. M. Angrisani and M. Clavelli, *Synthetic approaches to problems of fixed points in metric spaces*, Ann. Mat. Pura Appl. **170** (1996), 1–12.
3. F. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.
4. F. Browder, *Remarks on fixed point theorems of contractive type*, Nonlinear Anal. **3** (1979), 657–661.
5. F. Browder, *Erratum to the paper "Remarks on fixed point theorems of contractive type"*, Nonlinear Anal. **5** (1981), 111.
6. M. Hegedüs, *New generalizations of Banach's contraction principle*, Acta Sci. Math. (Szeged) **42** (1980), 87–89.
7. M. Hegedüs and T. Szilágyi, *Equivalent conditions and a new type of mappings*, Math. Japon. **25** (1980), 147–157.
8. J. Jachymski, *Order-theoretic aspects of metric fixed point theory*, in *Handbook of Metric Fixed Point Theory* (W. A. Kirk and B. Sims, eds.), Springer, 2001, pp. ??–??.
9. J. Jachymski, *Around Browder's fixed point theorem for contractions*, J. Fixed Point Theory Appl. **5** (2009), 47–61.
10. S. Kasahara, *A fixed point theorem of Meir–Keeler type*, Math. Sem. Notes **8** (1980), 131–135.
11. W. A. Kirk and L. M. Saliga, *Some results on existence and approximation in metric fixed point theory*, J. Comput. Appl. Math. **113** (2000), 141–152.
12. P. Kumam, N. V. Dung and K. Sitytithakerngkiet, *A generalization of Ćirić fixed point theorems*, Filomat **29** (2015), 1549–1556.
13. R. Pant and D. Khantwal, *Fixed points of single and multivalued mappings in metric spaces with applications*, Preprint.

14. S. Park, *On general contractive type conditions*, J. Korean Math. Soc. **17** (1980), 131–140.
15. S. Park, *Remarks on F. E. Browder's fixed point theorems of contractive type*, Proc. Coll. Natur. Sci. SNU **6** (1981), 47–51.
16. P. D. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal. **64** (2006), 546–557.
17. B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1978), 257–290.
18. M. R. Tasković, *Some results in the fixed point theory II*, Publ. Inst. Math. (Beograd) **41** (1987), 249–258.
19. M. R. Tasković, *On a result of W. A. Kirk and L. M. Saliga*, Mathematica Moravica **13** (2009), 117–119.
20. W. Walter, *Remarks on a paper by F. Browder about contraction*, Nonlinear Anal. **5** (1981), 21–25.