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SOLVABILITY OF EQUATIONS INVOLVING PERTURBATIONS OF m-ACCRETIVE OPERATORS IN BANACH SPACES

YUKIO TAKEUCHI[1] AND MASASHI TOYODA*[2]

[1] Address Takahashi Institute for Nonlinear Analysis, 1-11-11 Nakazato, Minami, Yokohama 232-0063, Japan

[2] Address Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan

ABSTRACT. It is purpose of this paper to give several results for the solvability of the equation $p \in Ax + Sx$, where A is an m-accretive operator on a Banach space E and S is a mapping on a subset of E, with elementary proofs. We give proofs of them without using degree theory.

KEYWORDS: Nonlinear operator, accretive operator, fixed point.
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1. Introduction

Let E be a real Banach space, A an m-accretive operator on E, S a mapping of a subset of E into E, and p an element of E. In [5], [6] and [8], the solvability of the equation

$$p \in Au + Su \tag{1.1}$$

for the case that $(I+A)^{-1}$ is compact and S is continuous has been studied. Results in these papers are proved using degree theory. For example, Theorem 1 in [5] is proved using Theorem 6.3.2 in [7] and Theorem 5 in [6] is proved using Theorem 4.4.11 in [7].

It is purpose of this paper to give several results for the solvability of the equation (1.1), with elementary proofs. We give proofs of them without using degree theory.

In Section 3, we introduce a fixed point theorem for a continuous mapping of a closed ball into E; see Proposition 3.1. To use Proposition 3.1 in proofs of main results, we need Propositions 2.2 and 2.3. In Section 4, using Propositions 2.2 and 3.1, we consider the solvability of the equation (1.1) in the case that $(I + A)^{-1}$ is compact and S is continuous. Moreover, in Section 5, using Propositions 2.3 and

* *Corresponding author.*
Email address : aho314159@yahoo.co.jp (Y. Takeuchi), mss-toyoda@is.sci.toho-u.ac.jp (M. Toyoda).
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3.1, we consider the equation related with (1.1) for the case that S is compact and the compactness of $(I+A)^{-1}$ is not required.

2. Preliminaries

Throughout this paper, E denotes a real Banach space with norm $\|\cdot\|$, E^* the topological dual of E and $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$. The normalized duality mapping of E is denoted by J, that is, it is a set-valued mapping of E into E^* defined by $Jx = \{x^* \in E^* \mid \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for $x \in E$.

For each subset C of E, we denote by $\overline{C}$ and ∂C, the closure of C and the boundary of C, respectively. Moreover we denote by $\operatorname{co} C$ the convex hull of C and by $\overline{\operatorname{co}}\, C$ the closed convex hull of C. Mazur's theorem asserts that if a subset C of E is compact, then $\overline{\operatorname{co}}\, C$ is compact; see Theorem 5.2.6 in [2].

Let x be an element of E and r a positive real number. We denote by $B_r(x)$ the open ball with center at x and radius r and by $B_r[x]$ the closed ball with center at x and radius r. To prove our main results, we need the following lemma related with the Minkowski functional associated to $B_r[0]$.

Lemma 2.1. *Let r be a positive real number. Define mappings f and M on E by*

$$f(x) = \frac{r}{\max\{r, \|x\|\}} \quad \text{and} \quad Mx = f(x)x$$

for $x \in E$. Then the following hold:

(1) *For $x \in B_r[0]$, $f(x) = 1$ and $Mx \in B_r[0]$;*
(2) *for $x \notin B_r[0]$, $f(x) \in (0, 1)$ and $Mx \in B_r[0]$;*
(3) *f and M are continuous.*

Therefore, f is a continuous mapping of E into $(0, 1]$ and M is a continuous mapping of E into $B_r[0]$.

Proof. We only show that the range of M is a subset of $B_r[0]$. If $x \in B_r[0]$, then $Mx = x \in B_r[0]$. If $x \notin B_r[0]$, then $\|Mx\| = \|\frac{r}{\|x\|}x\| = r$. Therefore, for all $x \in E$, we have $Mx \in B_r[0]$. □

Let T be a mapping of a subset C of E into E. We often use $D(T)$ instead of C. T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Nonexpansive mappings are continuous. T is said to be bounded if T maps bounded subsets of C onto bounded sets. Especially, nonexpansive mappings are bounded. T is said to be compact if T is continuous and it maps bounded subsets of C onto relatively compact sets. For continuous mappings, we know Brouwer's fixed point theorem; if T is a continuous mapping of a compact convex subset of an Euclidean space into itself, then T has a fixed point; see Theorem 2.1.11 in [11]. For an elementary proof of the theorem, see [13]. Schauder's fixed point theorem is the following; if T is a continuous mapping of a compact convex subset of a normed space into itself, then T has a fixed point; see Theorem 2.3.7 in [11].

Let A be a set-valued operator on E. The symbols $D(A)$ and $R(A)$ denote the domain and the range of A, respectively, that is, $D(A) = \{x \in E \mid Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Ax \mid x \in D(A)\}$. A is said to be accretive if for $x, y \in D(A)$, $u \in Ax$ and $v \in Ay$, there exists $j \in J(x - y)$ such that $\langle u - v, j\rangle \ge 0$. A is said to be m-accretive if A is accretive and $R(I + \lambda A) = E$ for all $\lambda > 0$.

Let A be an accretive operator on E and λ a positive real number. Since A is accretive, $(I + \lambda A)^{-1}$ is a single-valued mapping of $R(I + \lambda A)$ onto $D(A)$. The mapping $(I+\lambda A)^{-1}$ is called a resolvent of A and it is denoted by J_λ, that is, $J_\lambda x = (I+\lambda A)^{-1}x$ for $x \in R(I+\lambda A)$. Then, since $x \in J_\lambda x + \lambda A J_\lambda x$, we have

$$\frac{x - J_\lambda x}{\lambda} \in A J_\lambda x.$$

The mapping $\frac{I-J_\lambda}{\lambda}$ is called a Yosida approximation of A and it is denoted by A_λ, that is, $A_\lambda x = \frac{1}{\lambda}(x - J_\lambda x)$ for $x \in R(I+\lambda A)$. Therefore, for $x \in R(I+\lambda A)$, we have $A_\lambda x \in A J_\lambda x$. Since A is accretive, the resolvent J_λ is nonexpansive. Then, the resolvent is continuous and bounded. For these results, we refer the reader to [12].

In Sections 4 and 5, we use the following.

Proposition 2.2. *Let A be an m-accretive operator on E and S a continuous mapping of $D(S)$ into E such that J_1 is compact and $\overline{D(A)} \subset D(S)$. Let p be an element of E. Define a mapping T on E by*

$$Tx = p + J_1 x - S J_1 x$$

for $x \in E$. Then T is compact. Moreover if x is a fixed point of T, then $J_1 x$ is a solution of the equation (1.1).

Proof. Since J_1 is continuous, T is continuous. Since J_1 is compact, $\overline{J_1(B_r[0])}$ is compact for $r > 0$. Note that $J_1(B_r[0]) \subset \overline{J_1(B_r[0])} \subset \overline{D(A)} \subset D(S)$. Since S is continuous, $S(\overline{J_1(B_r[0])})$ is compact, that is, $SJ_1(B_r[0])$ is relatively compact. Therefore $T(B_r[0])$ is also relatively compact. Hence T is compact.

If x is a fixed point of T, then, since $p + J_1 x - SJ_1 x = x$, we have

$$p \quad = \quad x - J_1 x + SJ_1 x = A_1 x + SJ_1 x \in AJ_1 x + SJ_1 x.$$

Hence $J_1 x$ is a solution of the equation (1.1). □

Proposition 2.3. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E such that $D(A) \subset D(S)$. Let p be an element of E and n a positive integer. Define a mapping T_n on E by*

$$T_n x = p - SJ_n(nx)$$

for $x \in E$. Then T_n is compact. Moreover if x_n is a fixed point of T_n, then $J_n(nx_n)$ is a solution of the equation

$$p \in Au + Su + \frac{1}{n}u. \tag{2.1}$$

Proof. Since the resolvent J_n is continuous, T_n is continuous. Since J_n is bounded, $J_n(nB_r[0])$ is bounded for $r > 0$. Then, since S is compact, $SJ_n(nB_r[0])$ is relatively compact. Therefore $T_n(B_r[0])$ is also relatively compact. Hence T_n is compact.

If x_n is a fixed point of T_n, then, since $A_n(nx_n) = x_n - \frac{1}{n}J_n(nx_n)$, we have

$$\begin{aligned} p \quad &= \quad x_n + SJ_n(nx_n) \\ &= \quad A_n(nx_n) + SJ_n(nx_n) + \frac{1}{n}J_n(nx_n) \in AJ_n(nx_n) + SJ_n(nx_n) + \frac{1}{n}J_n(nx_n). \end{aligned}$$

Hence $J_n(nx_n)$ is a solution of the equation (2.1). □

In Section 3, we introduce a fixed point theorem for a continuous mapping of a closed ball into E; see Proposition 3.1. To prove Proposition 3.1, we need the following; see Lemma 1 in [10] and a result of Singbal in [1]. For the sake of completeness, we give a proof.

Lemma 2.4. *Let T be a continuous mapping of a closed ball $B_r[0]$ in E into itself. If $T(B_r[0])$ is relatively compact, then there exists a fixed point of T.*

Proof. Since $T(B_r[0])$ is relatively compact, $D = \overline{T(B_r[0])}$ is compact. For $s > 0$, there exist $x_1, x_2 \dots, x_n \in D$ such that $\{B_s(x_i) \mid i = 1, 2, \dots, n\}$ is a cover of D. For each $i = 1, 2, \dots, n$, we define a continuous mapping d_i on E by

$$d_i(x) = \max\{0, s - \|x - x_i\|\}$$

for $x \in E$. For any i and $x \in E$, $d_i(x)\|x - x_i\| \leq s d_i(x)$ holds because $d_i(x) \neq 0$ implies $\|x - x_i\| < s$. Moreover, for any $x \in D$, there exists i_0 satisfying $\|x - x_{i_0}\| < s$, that is, $d_{i_0}(x) > 0$. Therefore, for any $x \in D$, we have $\sum_{i=1}^{n} d_i(x) > 0$. If we define a function h_i for each $i = 1, 2, \dots, n$ by

$$h_i(x) = \frac{d_i(x)}{\sum_{i=1}^{n} d_i(x)}$$

for $x \in D$, then the following hold; for any $x \in D$ and $i = 1, 2, \dots, n$, $0 \leq h_i(x) \leq 1$; for any $x \in D$, $\sum_{i=1}^{n} h_i(x) = 1$. We consider a continuous mapping T_s on D defined by

$$T_s x = \sum_{i=1}^{n} h_i(x) x_i$$

for $x \in D$. Then we have

$$\|x - T_s x\| \leq \frac{1}{\sum_{i=1}^{n} d_i(x)} \sum_{i=1}^{n} d_i(x)\|x - x_i\| \leq \frac{s}{\sum_{i=1}^{n} d_i(x)} \sum_{i=1}^{n} d_i(x) = s$$

for $x \in D$. Since T is continuous, $T_s T$ is a continuous mapping of $\operatorname{co}(\{x_i\}_{i=1}^{n})$ into itself. By Brouwer's fixed point theorem, there exists $y \in \operatorname{co}(\{x_i\}_{i=1}^{n})$ satisfying $T_s T y = y$. Since $Ty \in D$, we have $\|Ty - T_s Ty\| \leq s$, that is, $\|Ty - y\| \leq s$. Therefore, for any $s > 0$, there exists $y \in \operatorname{co}(\{x_i\}_{i=1}^{n}) \subset B_r[0]$ such that $\|y - Ty\| \leq s$.

By the argument so far, there exists a sequence $\{y_m\} \subset B_r[0]$ satisfying $\|y_m - Ty_m\| \leq \frac{1}{m}$ for all $m = 1, 2, \dots$. That is, $\lim_{m \longrightarrow \infty} \|y_m - Ty_m\| = 0$. Since D is compact, there exists a subsequence $\{y_{m_j}\}$ of $\{y_m\}$ such that $\{Ty_{m_j}\}$ converges to some $z \in D$. Since $\lim_{m \longrightarrow \infty} \|y_m - Ty_m\| = 0$, $\{y_{m_j}\}$ also converges to z. Moreover, since T is continuous and

$$\|z - Tz\| \leq \|z - Ty_{m_j}\| + \|Ty_{m_j} - Tz\|,$$

we have $z = Tz$. □

Remark 2.5. Using Schauder's fixed point theorem, we can prove Lemma 2.4. In fact, since $\overline{T(B_r[0])}$ is compact, the closed convex hull of $\overline{T(B_r[0])}$ is also compact by Mazur's theorem. Then, by Schauder's fixed point theorem, we obtain the conclusion; see Theorem 4.4.10 in [7].

On the other hand, in the proof of Lemma 2.4, we use Brouwer's fixed point theorem. Consider the finite dimensional linear space L spanned by $\{x_i\}_{i=1}^{n}$, where $x_1, x_2, \dots, x_n$ are the elements in the proof of Lemma 2.4. Since any two Hausdorff linear topologies on L coincide, the relative topology of L induced by E is the Euclidian topology of L. Then we can consider that $\operatorname{co}(\{x_i\}_{i=1}^{n})$ is a compact convex subset of the Euclidean space L. By Brouwer's fixed point theorem, the continuous mapping $T_s T$ in the proof of Lemma 2.4 has a fixed point. Therefore the above proof of Lemma 2.4 is a direct proof from Brouwer's fixed point theorem. For the mapping T_s, see Theorem 2 in [9].

3. Fixed point theorems

To prove results in Sections 4 and 5, we need the following fixed point theorem.

Proposition 3.1. *Let T be a continuous mapping of a closed ball $B_r[0]$ into E such that $T(B_r[0])$ is relatively compact. Then, there exists $x \in B_r[0]$ such that*

$$f(Tx)Tx = x,$$

where f is defined as in Lemma 2.1. Moreover the following hold:
- (i) *If $Tx \in B_r[0]$, then $Tx = x$;*
- (ii) *if $x \in B_r(0)$, then $Tx = x$.*

Proof. Define a mapping V on $B_r[0]$ by $Vx = f(Tx)Tx$ for $x \in B_r[0]$. We know that the mapping M in Lemma 1 is continuous and the range of M is a subset of $B_r[0]$. Then we see that $M(\overline{T(B_r[0])})$ is compact and

$$V(B_r[0]) = MT(B_r[0]) \subset M(\overline{T(B_r[0])}) \subset B_r[0].$$

By Lemma 2.1, V is a continuous mapping of $B_r[0]$ into itself. Since $V(B_r[0])$ is relatively compact, by Lemma 2.4, there exists $x \in B_r[0]$ such that

$$Vx = x.$$

We show (i). If $Tx \in B_r[0]$, then we have $f(Tx) = 1$ by Lemma 2.1. Hence we have $Tx = x$. To prove (ii), suppose that $Tx \notin B_r[0]$. Then, since $x \in B_r(0)$ and $f(Tx)Tx = x$, we have

$$r > \|x\| = \|f(Tx)Tx\| = \left\| \frac{r}{\|Tx\|} Tx \right\| = \frac{r}{\|Tx\|} \|Tx\| = r.$$

This is a contradiction. Therefore $Tx \in B_r[0]$. Hence, by (i), $Tx = x$. □

The condition $Tx \in B_r[0]$ in (i) of Proposition 3.1 is related to the condition that $T(\partial B_r[0]) \subset B_r[0]$, which is a condition of Rothe's fixed point theorem; see Theorem 4.2.3 in [11].

Corollary 3.2. *Let T be a compact mapping of a closed ball $B_r[0]$ into E such that $T(\partial B_r[0]) \subset B_r[0]$. Then there exists a fixed point of T.*

Proof. By Proposition 3.1, there exists $x \in B_r[0]$ such that

$$f(Tx)Tx = x,$$

where f is defined as in Lemma 2.1. If $x \in B_r(0)$, then $Tx = x$ by (ii) of Proposition 3.1. If $x \in \partial B_r[0]$, then, since $T(\partial B_r[0]) \subset B_r[0]$, we have $Tx \in B_r[0]$. Hence $Tx = x$ by (i) of Theorem 3.1. □

The condition $Tx \in B_r[0]$ in (i) of Proposition 3.1 is related to the following:

$$\text{If } x \in \partial B_r[0] \text{ and } c > 1, \text{ then } Tx \neq cx. \tag{3.1}$$

For (3.1), see Theorem 0.2.3 in [3] and Theorems 4.4.3 and 6.3.2 in [7].

Corollary 3.3. *Let T be a continuous mapping of a closed ball $B_r[0]$ into E such that $T(B_r[0])$ is relatively compact and the condition (3.1) holds. Then there exists a fixed point of T.*

Proof. By Proposition 3.1, there exists $x \in B_r[0]$ such that $f(Tx)Tx = x$, where f is defined as in Lemma 2.1. Since $f(Tx)$ is in $(0,1]$ by Lemma 2.1, we have $c = \frac{1}{f(Tx)} \in [1, \infty)$ and

$$Tx = cx.$$

If $x \in \partial B_r[0]$, then we have $c = 1$ by the condition (3.1). Hence $Tx = x$. If $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. □

4. Main results

In this section, we consider the solvability of the equation (1.1) for the case that $(I + A)^{-1}$ is compact and S is continuous. Proposition 3.1 is crucial in the proofs of results in this section.

By Propositions 2.2 and 3.1, we obtain the following. The condition (4.1) is related to the condition $T(\partial B_r[0]) \subset B_r[0]$ in Corollary 3.2.

Theorem 4.1. *Let A be an m-accretive operator on E such that J_1 is compact and S is a continuous mapping of $D(S)$ into E with $\overline{D(A)} \subset D(S)$. Let p be an element of E. Suppose there exists a positive constant r satisfying the following:*

$$\text{If } x \in \partial B_r[0], \text{ then } \|p + J_1x - SJ_1x\| \leq r. \tag{4.1}$$

Then the equation (1.1) has a solution.

Proof. Let T be a mapping defined by $Tx = p + J_1x - SJ_1x$ for $x \in E$. By Proposition 2.2, T is compact. By Proposition 3.1, there exists $x \in B_r[0]$ such that

$$f(Tx)Tx = x,$$

where f is defined as in Lemma 2.1.

For the case that $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. For the case that $x \in \partial B_r[0]$, since $\|p + J_1x - SJ_1x\| \leq r$, we have $\|Tx\| \leq r$, that is, $Tx \in B_r[0]$. By (i) of Proposition 3.1, we see $Tx = x$. Therefore, in both cases, $Tx = x$ holds. Hence, by Proposition 2.2, (1.1) has a solution. □

Theorem 4.2. *Let A be an m-accretive operator on E such that J_1 is compact and S is a continuous mapping of $D(S)$ into E with $\overline{D(A)} \subset D(S)$. Let p be an element of E. Suppose there exists a positive constant r such that if $x \in \partial B_r[0]$, then there exists $j \in E^*$ satisfying $\langle x, j\rangle > 0$ and*

$$\langle A_1x - p + SJ_1x, j\rangle \geq 0.$$

Then the equation (1.1) has a solution.

Proof. Let T be a mapping defined by $Tx = p + J_1x - SJ_1x$ for $x \in E$. By Proposition 2.2, T is compact. By Proposition 3.1, there exists $x \in B_r[0]$ such that $f(Tx)Tx = x$, where f is defined as in Lemma 2.1. Since $f(Tx) \in (0, 1]$, by Lemma 2.1, we have $1 \leq c = \frac{1}{f(Tx)}$ and

$$Tx = cx.$$

For the case that $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. For the case that $x \in \partial B_r[0]$, there exists $j \in E^*$ such that $\langle x, j\rangle > 0$ and $\langle A_1x - p + SJ_1x, j\rangle \geq 0$. Since $A_1x - p + SJ_1x = x - Tx = x - cx$, we have

$$0 \leq \langle x, j\rangle\langle A_1x - p + SJ_1x, j\rangle = \langle x, j\rangle\langle x - cx, j\rangle = (1 - c)\langle x, j\rangle^2.$$

Then, since $\langle x, j\rangle \neq 0$, we have $c \leq 1$. Thus $c = 1$. Therefore, in both cases, $Tx = x$ holds. Hence, by Proposition 2.2, (1.1) has a solution. □

Moreover, we obtain the following.

Theorem 4.3. *Let A be an m-accretive operator on E such that J_1 is compact and S is a continuous mapping of $D(S)$ into E with $\overline{D(A)} \subset D(S)$. Let p be an element of E. Suppose there exist positive constant b and r which satisfy $B_b(0) \cap D(A) \neq \emptyset$, $p \in B_b(0)$,*

$$r > 2b + \sup\{\|Sx\| \mid x \in B_b(0) \cap D(A)\},$$

and the following: If $x \in \partial B_r[0]$ satisfies $J_1 x \notin B_b(0)$, then there exists $j \in E^*$ satisfying $\langle x, j\rangle > 0$ and

$$\langle A_1 x - p + SJ_1 x, j\rangle \geq 0.$$

Then the equation (1.1) has a solution.

Proof. Let T be a mapping defined by $Tx = p + J_1 x - SJ_1 x$ for $x \in E$. By Proposition 2.2, T is compact. By Proposition 3.1, there exists $x \in B_r[0]$ such that $f(Tx)Tx = x$, where f is defined as in Lemma 2.1. Since $f(Tx) \in (0, 1]$, by Lemma 2.1, we see $1 \leq c = \frac{1}{f(Tx)}$ and $Tx = cx$.

For the case that $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. Next we consider the case that $x \in \partial B_r[0]$. Assume that $J_1 x \in B_b(0)$. Then, by $c \in [1, \infty)$, we have

$$\begin{aligned} r &\leq cr = c\|x\| = \|cx\| = \|p + J_1 x - SJ_1 x\| \\ &\leq \|p\| + \|J_1 x\| + \|SJ_1 x\| < 2b + \sup\{\|Sx\| \mid x \in D(A) \cap B_b(0)\} < r. \end{aligned}$$

This is a contradiction. So, $J_1 x \notin B_b(0)$ holds. From this, there exists $j \in E^*$ satisfying $\langle x, j\rangle > 0$ and $\langle A_1 x - p + SJ_1 x, j\rangle \geq 0$. Then, in the same way as in the proof of Theorem 4.2, we have $c = 1$. Therefore, in both cases, $Tx = x$ holds. Hence, by Proposition 2.2, (1.1) has a solution. □

By Theorem 4.3, we obtain the following. Theorem 2 in [5] is related to the condition (4.2).

Corollary 4.1. *Let A be an m-accretive operator on E such that J_1 is compact and S is a bounded continuous mapping of $\overline{D(A)}$ into E. Let p be an element of E. Suppose there exists a positive constant b which satisfy $J_1 0 \in B_b(0)$, $p \in B_b(0)$ and the following:*

$$\text{If } z \in D(A), \|z\| \geq b,\ y \in Az \text{ and } j \in J(z - J_1 0),\ \text{then } \langle y - p + Sz, j\rangle \geq 0. \quad (4.2)$$

Then the equation (1.1) has a solution.

Proof. Note that, since $J_1 0 \in D(A) \cap B_b(0)$, $D(A) \cap B_b(0) \neq \emptyset$. Since S is bounded, there exists $r > 0$ satisfying the condition $r > 2b + \sup\{\|Sx\| \mid x \in B_b(0) \cap D(A)\}$ in Theorem 4.3. Let x be an element of $\partial B_r[0]$ with $J_1 x \notin B_b(0)$. Set $z = J_1 x$. Obviously, we see $z \in D(A)$, $\|z\| \geq b$ and $A_1 x \in AJ_1 x = Az$. Then, since $A_1 x \in Az$, $A_1 0 \in AJ_1 0$ and A is accretive, there exists $j \in J(z - J_1 0) = J(J_1 x - J_1 0)$ such that

$$\langle A_1 x - A_1 0, j\rangle \geq 0.$$

By (4.2), for such z, $A_1 x$ and j, $\langle A_1 x - p + Sz, j\rangle = \langle A_1 x - p + SJ_1 x, j\rangle \geq 0$ holds. Furthermore, since $J_1 0 \in B_b(0)$ and $J_1 x \notin B_b(0)$, we have

$$\begin{aligned} \langle x, j\rangle &= \langle A_1 x + J_1 x, j\rangle \\ &= \langle A_1 x - A_1 0, j\rangle + \langle A_1 0 + J_1 x, j\rangle \\ &\geq \langle A_1 0 + J_1 x, j\rangle \\ &= \langle J_1 x - J_1 0, j\rangle = \|J_1 x - J_1 0\|^2 > 0. \end{aligned}$$

So, all conditions of Theorem 4.3 are fulfilled. That is, (1.1) has a solution. □

5. Results for the case that S is compact

In this section, we consider

$$p \in \overline{(A+S)(B_c(0) \cap D(A))}, \tag{5.1}$$

where A is an m-accretive operator on E, S is a compact mapping on a subset of E, p is an element of E and c is a positive constant. Theorem 1 in [4] and Theorem 3 in [6] are related to (5.1).

By Propositions 2.3 and 3.1, we obtain the following results.

Theorem 5.1. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E with $D(A) \subset D(S)$. Let p be an element of E. Suppose there exist positive constants b, r and a positive integer n_0 which satisfy $p \in B_b(0)$, $J_n 0 \in B_b(0)$ for all positive integers n,*

$$r > b + \sup\{\|Sx\| \mid x \in B_{2b}(0) \cap D(A)\},$$

and the following: If $n \geq n_0$, $x \in B_r[0]$ and $J_n(nx) \notin B_{2b}(0)$, then for all $j_n \in J(J_n(nx) - J_n 0)$,

$$\langle A_n(nx) - p + SJ_n(nx), j_n \rangle \geq 0.$$

Then the equation (5.1) has a solution as $c = 2b$.

Proof. Let n be a positive integer with $n \geq n_0$ and T_n a mapping defined by $T_n x = p - SJ_n(nx)$ for $x \in E$. By Proposition 2.3, T_n is compact. By Proposition 3.1, there exists $x_n \in B_r[0]$ such that

$$f(T_n x_n) T_n x_n = x_n,$$

where f is defined as in Lemma 2.1. Since $f(T_n x_n) \in (0, 1]$ by Lemma 2.1, we have $1 \leq c_n = \frac{1}{f(T_n x_n)}$ and

$$T_n x_n = c_n x_n.$$

We show that $J_n(nx_n) \in B_{2b}(0)$. Assume that $J_n(nx_n) \notin B_{2b}(0)$. Since $J_n 0 \in B_b(0)$, we have

$$\|J_n(nx_n) - J_n 0\| \quad \geq \quad \|J_n(nx_n) - J_n 0\| - \|J_n 0\| \geq \|J_n(nx_n)\| - 2\|J_n 0\| > 0.$$

So, we see $\|J_n(nx_n) - J_n 0\| > 0$ and $s = \|J_n(nx_n)\| - 2\|J_n 0\| > 0$.

Since $A_n(nx_n) \in AJ_n(nx_n)$, $A_n 0 \in AJ_n 0$ and A is accretive, there exists $j_n \in J(J_n(nx_n) - J_n 0)$ satisfying $\langle A_n(nx_n) - A_n 0, j_n \rangle \geq 0$. Then, since

$$\left\langle \frac{1}{n} J_n(nx_n) + A_n 0, j_n \right\rangle = \frac{1}{n} \langle J_n(nx_n) - J_n 0, j_n \rangle = \frac{1}{n} \|J_n(nx_n) - J_n 0\|^2$$

and $A_n(nx_n) = \frac{1}{n}(nx_n - J_n(nx_n))$, we see

$$\begin{aligned} \langle x_n, j_n \rangle &= \left\langle A_n(nx_n) + \frac{1}{n} J_n(nx_n), j_n \right\rangle \\ &= \left\langle \frac{1}{n} J_n(nx_n) + A_n 0, j_n \right\rangle + \langle A_n(nx_n) - A_n 0, j_n \rangle \\ &\geq \frac{1}{n} \|J_n(nx_n) - J_n 0\|^2. \end{aligned}$$

So we have $\langle x_n, j_n \rangle \geq \frac{1}{n} \|J_n(nx_n) - J_n 0\|^2 > 0$. Also, we see

$$\begin{aligned} & - \left\langle \frac{1}{n} J_n(nx_n) - \frac{1}{n} J_n 0, j_n \right\rangle - \frac{1}{n} \langle J_n 0, j_n \rangle \\ \leq \; & -\frac{1}{n} \|J_n(nx_n) - J_n 0\|^2 + \frac{1}{n} \|J_n 0\| \|J_n(nx_n) - J_n 0\| \end{aligned}$$

$$\begin{aligned} &\leq -\frac{1}{n}\|J_n(nx_n) - J_n0\| \left(\|J_n(nx_n) - J_n0\| - \|J_n0\|\right) \\ &\leq -\frac{s}{n}\|J_n(nx_n) - J_n0\|. \end{aligned}$$

By the argument so far, considering $n \geq n_0$, $x \in B_r[0]$, $J_n(nx) \notin B_{2b}(0)$, and $j_n \in J(J_n(nx_n) - J_n0)$, the following holds:

$$\begin{aligned} 0 &\leq \langle A_n(nx_n) - p + SJ_n(nx_n), j_n\rangle \\ &= \langle x_n - p + SJ_n(nx_n), j_n\rangle - \left\langle \frac{1}{n}J_n(nx_n) - \frac{1}{n}J_n0, j_n \right\rangle - \frac{1}{n}\langle J_n0, j_n\rangle \\ &\leq (1 - c_n)\langle x_n, j_n\rangle - \frac{s}{n}\|J_n(nx_n) - J_n0\|. \end{aligned}$$

Furthermore, since $\|J_n(nx_n) - J_n0\| > 0$, $s > 0$, $\langle x_n, j_n\rangle > 0$ and $1 \leq c_n$, we have

$$0 < \frac{s}{n}\|J_n(nx_n) - J_n0\| \leq (1 - c_n)\langle x_n, j_n\rangle \leq 0.$$

This is a contradiction. Hence $J_n(nx_n) \in B_{2b}(0)$ holds.

We show that $x_n \in B_r(0)$. Assume that $x_n \in \partial B_r[0]$. Then, since $J_n(nx_n) \in B_{2b}(0)$ and $1 \leq c_n$, we have a contradiction:

$$\begin{aligned} r &\leq c_n\|x_n\| = \|c_nx_n\| = \|p - SJ_n(nx_n)\| \leq \|p\| + \|SJ_n(nx_n)\| \\ &\leq b + \|SJ_n(nx_n)\| < r. \end{aligned}$$

Thus, since $x_n \in B_r(0)$, by (ii) of Proposition 3.1, $T_nx_n = x_n$ holds.

Since $n \geq n_0$ is arbitrary, by Proposition 2.3, there is a sequence $\{x_n\}_{n\geq n_0}$ such that $J_n(nx_n)$ is a solution of (2.1) for each $n \geq n_0$, that is,

$$\begin{aligned} p - \frac{1}{n}J_n(nx_n) &= A_n(nx_n) + SJ_n(nx_n) \\ &\in AJ_n(nx_n) + SJ_n(nx_n) \in (A + S)(B_{2b}(0) \cap D(A)) \end{aligned}$$

for all $n \geq n_0$. Since $J_n(nx_n) \in B_{2b}(0)$, we know $\lim_{n\longrightarrow\infty} \left\|\frac{1}{n}J_n(nx_n)\right\| = 0$. Therefore, we have the desired result $p \in \overline{(A + S)(B_{2b}(0) \cap D(A))}$. □

Theorem 5.2. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E with $D(A) \subset D(S)$. Let p be an element of E. Suppose there exist a positive constant r and a positive integer n_0 which satisfy the following:*

$$\text{If } n \geq n_0 \text{ and } x \in \partial B_r[0], \text{ then } \|p - SJ_n(nx)\| \leq r.$$

Then there exists a sequence $\{x_n\}_{n\geq n_0}$ such that $J_n(nx_n)$ is a solution of the equation (2.1) for each $n \geq n_0$. Suppose further $\lim_{n\longrightarrow\infty} \left\|\frac{1}{n}J_n(nx_n)\right\| = 0$. Then, $p \in \overline{(A + S)(D(A))}$ holds.

Proof. Let n be a positive integer with $n \geq n_0$ and T_n a mapping defined by $T_nx = p - SJ_n(nx)$ for $x \in E$. By Proposition 2.3, T_n is compact. By Proposition 3.1, there exists $x_n \in B_r[0]$ such that $f(T_nx_n)T_nx_n = x_n$, where f is defined as in Lemma 2.1.

For the case that $x_n \in B_r(0)$, by (ii) of Proposition 3.1, we have $T_nx_n = x_n$. For the case that $x_n \in \partial B_r[0]$, we know $\|T_nx_n\| \leq r$, that is, $T_nx_n \in B_r[0]$. By (i) of Proposition 3.1, we have $T_nx_n = x_n$. In both cases, $T_nx_n = x_n$ holds.

Since $n \geq n_0$ is arbitrary, by Proposition 2.3, there is a sequence $\{x_n\}_{n\geq n_0}$ such that $J_n(nx_n)$ is a solution of (2.1) for each $n \geq n_0$. Since for all $n \geq n_0$,

$$p - \frac{1}{n}J_n(nx_n) = A_n(nx_n) + SJ_n(nx_n) \in (A + S)(D(A)),$$

in the case that $\lim_{n\longrightarrow\infty} \left\|\frac{1}{n}J_n(nx_n)\right\| = 0$, $p \in \overline{(A + S)(D(A))}$ holds. □

Theorem 5.3. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E with $D(A) \subset D(S)$. Let p be an element of E. Suppose there exist a positive constant r and a positive integer n_0 which satisfy the following: If $n \geq n_0$ and $x \in \partial B_r[0]$, then there exists $j_n \in E^*$ such that $\langle x, j_n \rangle > 0$ and*

$$\langle x - p + SJ_n(nx), j_n \rangle \geq 0.$$

Then there exists a sequence $\{x_n\}_{n \geq n_0}$ such that $J_n(nx_n)$ is a solution of the equation (2.1) for each $n \geq n_0$. Suppose further $\lim_{n \longrightarrow \infty} \left\| \frac{1}{n} J_n(nx_n) \right\| = 0$. Then, $p \in \overline{(A+S)(D(A))}$ holds.

Proof. Let n be a positive integer with $n \geq n_0$ and T_n a mapping defined by $T_n x = p - SJ_n(nx)$ for $x \in E$. By Proposition 2.3, T_n is compact. By Proposition 3.1, there exists $x_n \in B_r[0]$ such that $f(T_n x_n) T_n x_n = x_n$, where f is defined as in Lemma 2.1. Also, by Lemma 2.1, we know $f(T_n x_n) \in (0, 1]$ and $c_n = \frac{1}{f(T_n x_n)} \in [1, \infty)$. So, $T_n x_n = c_n x_n$.

For the case that $x_n \in B_r(0)$, by (ii) of Proposition 3.1, we have $T_n x_n = x_n$. For the case that $x_n \in \partial B_r[0]$, there exists $j_n \in E^*$ such that $\langle x_n, j_n \rangle > 0$ and $\langle x_n - p + SJ_n(nx_n), j_n \rangle \geq 0$. From this, we have

$$0 \leq \langle x_n, j_n \rangle \langle x_n - p + SJ_n(nx_n), j_n \rangle = (1 - c_n) \langle x_n, j_n \rangle^2.$$

Then we see $c_n = 1$ and $T_n x_n = x_n$. In both cases, $T_n x_n = x_n$ holds. The rest of the proof is the same as in the proof of Theorem 5.2. □

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