# A characterization of the identity function with some property on $GP$

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**Abstract**

We prove that if a multiplicative function $f$ satisfies

$$f(a+b+c) = f(a) + f(b) + f(c), \text{ for all } a, b, c \in GP.$$

Then $f$ is the identity function.

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## 1 Introduction

In 1992, Spiro [10] proved that if a multiplicative function $f$ satisfies $f(p_0) = 0$ for some prime $p_0$ and $f(p+q) = f(p) + f(q)$ for all primes $p$ and $q$, then $f$ is the identity function. More generally, Spiro asked which subset $E$ of $\mathbb{Z}^+$ could determine an arithmetic function $f$ uniquely in $S$ under conditions $f(a+b) = f(a) + f(b)$ for all $a, b \in E$, where $S$ is a set of arithmetic functions. Such a set $E$ is called an additive uniqueness set for $S$ following Spiro's theme. After Spiro's work, this interesting subject has been studied and extended in many directions (see [2] - [9]). In particular, Chung and Phong [4] showed that the set of all triangular numbers is an additive uniqueness set for multiplicative functions, while Chung [3] showed that the set of square numbers is not an additive uniqueness set for multiplicative functions.

The nonzero generalized pentagonal numbers (in brief, $GP$) are the integers obtained by the formula

$$P_n = \frac{n(3n-1)}{2}, n \in \mathbb{Z} \setminus \{0\}.$$

That is,

$$GP = \{P_1, P_{-1}, P_2, P_{-2}, P_3, P_{-3}, P_4, P_{-4}, P_5, ...\} = \{1, 2, 5, 7, 12, 15, 22, 26, 35, ...\}.$$

In 2011, Fang [6] proved that if $f$ is a multiplicative function such that there exists a prime $p_0$ at which $f$ does not vanish and $f$ satisfies the equation $f(p+q+r) = f(p) + f(q) + f(r)$ for all primes $p, q$ and $r$, then $f$ is an identity function for all integers $n \geq 1$.

In this article, we prove that if a multiplicative function $f$ satisfies $f(a+b+c) = f(a) + f(b) + f(c)$, for all $a, b, c \in GP$, then $f$ is the identity function.

## 2 Preliminaries

In this section, we collect a definition and a lemma which will be used in the following section.

**Definition 2.1.** An arithmetic function $f : \mathbb{Z}^+ \longrightarrow \mathbb{C}$ is called multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $m$ and $n$ are coprime.

**Lemma 2.2.** [1] Let $p \neq 5$ be a prime and let $r \in \mathbb{Z}^+$. Then there are $a, b \in GP$ and $\lambda \in \mathbb{Z}^+$ such that $\lambda p^r = a + b$, where $gcd(\lambda, p) = 1$ with $\lambda < p^r$. Moreover, $a$ and $b$ are products of coprime numbers which are smaller than $p^r$. Furthermore, the same statement is true for $p = 5$ with $r > 1$.

## 3 Main Results

We obtain the following two lemmas to help the proof of a main theorem.

**Lemma 3.1.** Let $x = a + b$ for some $a, b \in GP$ such that $a > 1$ or $b > 1$, then $x = a' + b' + c'$ for some $a', b', c' \in GP$.

*Proof.* Let $x = a + b$ for some $a, b \in GP$. Then there exist $n, n' \in \mathbb{Z} \setminus \{0\}$ such that

$$a = \frac{n(3n-1)}{2} \text{ and } b = \frac{n'(3n'-1)}{2}, \text{ and so } x = \frac{n(3n-1)}{2} + \frac{n'(3n'-1)}{2}.$$

Now, consider $a > 1$, we have

$$a = \frac{n_1(3n_1-1)}{2} + \frac{n_2(3n_2-1)}{2} \quad \text{for some } n_1, n_2 \in \mathbb{Z} \setminus \{0\}.$$

or

$$a \neq \frac{n_1(3n_1-1)}{2} + \frac{n_2(3n_2-1)}{2} \quad \text{for all } n_1, n_2 \in \mathbb{Z} \setminus \{0\}.$$

**Case 1** : If

$$a = \frac{n_1(3n_1-1)}{2} + \frac{n_2(3n_2-1)}{2} \quad \text{for some } n_1, n_2 \in \mathbb{Z} \setminus \{0\}.$$

Then

$$x = \frac{n_1(3n_1-1)}{2} + \frac{n_2(3n_2-1)}{2} + \frac{n'(3n'-1)}{2} \quad \text{for some } n_1, n_2, n' \in \mathbb{Z} \setminus \{0\}.$$

That is $x = a' + b' + c'$, where $a' = \frac{n_1(3n_1-1)}{2}$, $b' = \frac{n_2(3n_2-1)}{2}$ and $c' = \frac{n'(3n'-1)}{2}$.

**Case 2** : If

$$a \neq \frac{n_1(3n_1-1)}{2} + \frac{n_2(3n_2-1)}{2} \quad \text{for all } n_1, n_2 \in \mathbb{Z} \setminus \{0\}.$$

Since $a > 1$ and $x = a + b$, so $x > 5$. It follows that $x \geq 6$ and

$$x = \frac{n_1'(3n_1'-1)}{2} + \frac{n_2'(3n_2'-1)}{2} + \frac{n_3'(3n_3'-1)}{2} \quad \text{for some } n_1', n_2', n_3' \in \mathbb{Z} \setminus \{0\}.$$

That is $x = a' + b' + c'$, where $a' = \frac{n_1'(3n_1'-1)}{2}$, $b' = \frac{n_2'(3n_2'-1)}{2}$ and $c' = \frac{n_3'(3n_3'-1)}{2}$.

For $b > 1$, we can prove similar to $a > 1$. □

**Lemma 3.2.** Let $p$ be a prime and let $r \in \mathbb{Z}^+$. Then there are $a, b, c \in GP$ and $\lambda \in \mathbb{Z}^+$ such that $\lambda p^r = a + b + c$, where $gcd(\lambda, p) = 1$ with $\lambda < p^r$. Moreover, $a$, $b$ and $c$ are products of coprime numbers which are smaller than $p^r$.

*Proof.* By Lemma 2.2 and Lemma 3.1, it follows that the proof is completed. □

**Theorem 3.3.** If a multiplicative function $f$ satisfies

$$f(a+b+c) = f(a) + f(b) + f(c),$$

for all $a, b, c \in GP$, then $f$ is the identity function.

*Proof.* We will show that $f(n) = n$ for any positive integer $n$ and using the induction on $n$.

(1) By the multiplicative property of $f$, we get $f(1) = 1$.

(2) By the property of $f$ on $GP$, we get

$$\begin{aligned}
f(3) &= f(1) + f(1) + f(1) = 3, \\
f(4) &= f(2) + f(1) + f(1) = f(2) + 2, \\
f(5) &= f(2) + f(2) + f(1) = 2f(2) + 1, \\
f(6) &= f(2)f(3) = 3f(2), \\
f(7) &= f(5) + f(1) + f(1) = 7, \\
f(8) &= f(5) + f(2) + f(1) = 6 + f(2), \\
f(9) &= f(7) + f(1) + f(1) = 9, \\
f(10) &= f(7) + f(2) + f(1) = 8 + f(2), \\
f(11) &= f(7) + f(2) + f(2) = 7 + 2f(2), \\
f(12) &= f(9) + f(2) + f(1) = 10 + f(2).
\end{aligned}$$

Consider,

$$\begin{aligned}
10 + f(2) = f(12) &= f(3)f(4) = 3(2 + f(2)) = 6 + 3f(2), \\
f(2) &= 2
\end{aligned}$$

Hence, $f(4) = 4, f(5) = 5, f(6) = 6, f(8) = 8, f(10) = 10, f(11) = 11, f(12) = 12$.

(3) Let $m$ be an integer larger than 12. Suppose that $f(k) = k$ for all $k < m$.

The multiplicativity of $f$ and the factorization of $m = \prod_{i=1}^{l} p_i^{e_i}$. Then $f(m) = \prod_{i=1}^{l} f(p_i^{e_i})$.

If $\ell \geq 2$, then $p_i^{ei} < m$ for all $i$ and hence the induction hypothesis guarantees that $f(p_i^{ei}) = p_i^{ei}$.

$$\text{So} \quad f(m) = m.$$

If $\ell = 1$, then $m = p^e$ for some prime $p$ and a positive integer $e$. By Lemma 3.2, we have

$$\lambda p^e = a + b + c,$$

for some $a, b, c \in GP$ and $a < p^e, b < p^e, c < p^e$, $gcd(\lambda, p) = 1$. Then

$$f(\lambda p^e) = f(a + b + c)$$

$$f(\lambda)f(p^e) = f(a) + f(b) + f(c).$$

Since $f(\lambda) = \lambda$, $f(a) = a$, $f(b) = b$, $f(c) = c$, we have

$$\lambda f(m) = a + b + c$$

$$\lambda f(m) = \lambda p^e$$

$$f(m) = m.$$

By the mathematical induction, we have $f(n) = n$. Hence $f$ is the identity function. □

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