Strong Convergence Theorem for $k-$ strictly pseudo $\lambda-$ hybrid Mappings and Equilibrium Problem in Hilbert Spaces

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Abstract

In this paper we prove a strong convergence theorem for $k-$strictly pseudo $\lambda-$hybrid mappings and equilibrium problem in Hilbert spaces by using an idea of mean convergence. The main result of this paper extend the results obtained by Osilike and Isiogugu (Nonlinear Analysis 74 (2011) 1814-1822) and Kurokawa and Takahashi (Nonlinear Analysis 73 (2010) 1562-1568).

Keyword: Hilbert spaces: k-strictly pseudononspreading mappings: $\lambda-$hybrid mappings: fixed points: strong convergence

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1 Introduction

Let H be a real Hilbert space. A mapping $T : D(T) \subseteq H \longrightarrow H$ is said to be $L-Lipschitzian$ if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in D(T). \tag{1.1}$$

If $L < 1$ in (1.1), T is said to be *strictly contractive*, T is said to be $quasi - nonexpansive$ if $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for al x in $D(T)$ and for all p in $F(T)$. Furthermore, T is said to be $firmly\ nonexpansive$ if

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in D(T).$$

Every nonexpansive mapping with a nonempty fixed point set $F(T)$ is quasi-nonexpansive, and firmly nonexpansive mappings are important examples of nonexpansive mappings.[1]

[1]

In 2010, Kohsaka and Takahashi ([12],[13]) introduced an important class of mappings which they called the class of *nonspreading mappings*. Let E be a real smooth, strictly convex and reflexive Banach space, and let j denote the duality mapping of E. Let C be a nonempty closes convex subset of E. They called a mapping $T : C \to C$ *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$, $\forall x, y \in E$. They considered the class of nonspreading mappings to study the resolvents of a maximal monotone operators in the Banach space. This class of mappings is deduced from the class of firmly nonexpansive mappings. Observe that if E is a real Hilbert space, then j is the identity and

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \|x - y\|^2.$$

If C is a nonempty closed convex subset of a Hilbert space, then $T : C \longrightarrow C$ is *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C. \tag{1.2}$$

It is shown in ([11]) that (1.2) is equaivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{1.3}$$

Observe that if T is nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.

A mapping $T : C \longrightarrow H$ is called *hybrid* if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$.

In 2010, Osilike and Isiogugu ([16]) introduced a new mapping of nonspreading-type as follows. A mapping $T : D(T) \subseteq H \longrightarrow H$ is said to be is k-strictly pseudononspreading if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Let $\beta \in [k, 1)$ and $T_\beta = \beta I + (1 - \beta)T$. Then $F(T) = F(T_\beta)$.

Clearly every nonspreading mapping is k-strictly psedononspreading. For example shows that the class of k-strictly pseudononspreading mapping is more general than the class of nonspreading mappings (see example([16])).[2]

[2] *Academic Journal of Science and Applied Science 2017(1) January-June pp.119-129*

Observe that if T is k-strictly pseudononspreading and $F(T) \neq \emptyset$, then for all $x \in D(T)$ and for all $p \in F(T)$ we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2. \tag{1.4}$$

Thus every k-strictly pseudononspreading map with a nonempty fixed point set $F(T)$ is *demicontractive* (see example([7], [15]).

In 2010, Kohsaka and Takahashi ([12]) introduced a new class of mappings which is more general than a class of hybrid mappings. A mapping $T : C \to H$ is generalized hybrid if there are $\alpha, \beta \in R$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2,$$

for all $x, y \in C$.

Recently, S. Suantai defined a mapping $T : C \to C$ is said to be $k-strictlypseudo\ \lambda-hybrid$, if there exist $k \in [0, 1)$ and $\lambda \geq 0$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle + k\|(x - Tx) - (y - Ty)\|^2 \tag{1.5}$$

(i) If $k = 0$ and $\lambda = \frac{1}{2}$, then T is hybrid.

(ii) If $k = 0$ and $\lambda = 1$, then T is nonspreading.

(iii) If $\lambda = 1$, then T is $k-$ strictly pseudononspreading.

(iv) If $\lambda = 0$, then T is $k-$ strict pseudo-contractive.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0,\ \forall y \in C\}. \tag{1.6}$$

The set of generalized equilibrium problem is denoted by EP i.e.,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C\}$$

Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying

$(A1)$ $F(x, x) = 0 \quad \forall x \in C$;

$(A2)$ F is monotone, i.e. $F(x, y) + F(y, x) \leq 0\ \ \forall x, y \in C$;[3]

$(A3)$ $\forall x, y, z \in C$,

$\lim_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y)$;

$(A4)$ $\forall x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous; In this paper, using an idea of mean convergence, we prove a strong convergence theorem for $k-$strictly pseudo $\lambda-$hybrid mappings in a Hilbert space.

2 Preliminaries

Let E be real Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclased at a point $p \in D(T)$ (see example[6]) if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(T)$ which converges weakly to a point $x \in D(T)$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx = p$.

Lemma 2.1. ([16]) Let H be a real Hilbert space. Then the following well known results hold:

(1) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$,

for all $x, y \in H$ and for all $t \in [0, 1]$.

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$ for all $x, y \in H$.

(3) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to $z \in H$ then

$$\limsup_{n \longrightarrow \infty} \|x_n - y\|^2 = \limsup_{n \longrightarrow \infty} \|x_n - z\|^2 + \|z - y\|^2 \quad \forall y \in H.$$

Let C be nonempty closed convex subset of a real Hilbert space H. The nearest point projection $P_C : H \longrightarrow C$ defined from H onto C is the function which assigns to each $x \in H$ its nearest point denoted by $P_C x$ in C. Thus $P_C x$ is the unique point in C such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C.$$

It is known that for each $x \in H$

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \quad \forall y \in C.$$

Lemma 2.2. ([20]) Let C be nonempty closed convex subset of a real Hilbert space H. Let $P_C : H \longrightarrow C$ be the metric projection of H onto C. Let $\{x_n\}_{n=1}^{\infty}$ be sequence in C and let $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all u in C. Then $\{P_C x_n\}_{n=1}^{\infty}$ converges strongly.

Lemma 2.3. ([5]) Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\} \ \ for\, all \ \ x \in H. \tag{2.1}$$

Then the following hold:

(1) T_r is is single-valued;

(2) T_r is firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 2.4. ([1],[21]) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the condition

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \beta_n, \quad n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequence such that

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\limsup_{n \longrightarrow \infty} \beta_n \leq 0$.

Then $\lim_{n \longrightarrow \infty} a_n = 0$.

3 Main Results

Theorem 3.1. Let C be a nonempty closed convex subset of of a real Hilbert space H. Let $T : C \longrightarrow C$ be a k-strictly pseudo λ-hybrid mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1 - \beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ satisfying the conditions:

$$\lim_{n \longrightarrow \infty} \alpha_n = 0 \ \ and \ \ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \ \ n \geq 1, \\ z_n = \frac{1}{n} \sum_{k=0}^{n-1} T_\beta^k x_n, \ \ n \geq 1, \end{cases} \tag{3.1}$$

[5] Then $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converges strongly to $P_{F(T)}u$, where $P_{F(T)} : H \longrightarrow F(T)$ is the metric projection of H onto $F(T)$.

Proof. Let $T_\beta x := \beta x + (1-\beta)Tx$. Then for all $x, y \in C$ we have

$$\begin{aligned}
\|T_\beta x - T_\beta y\|^2 &= \|\beta x + (1-\beta)Tx - \beta y - (1-\beta)Ty\|^2 \\
&= \|\beta(x-y) + (1-\beta)(Tx-Ty)\|^2 \\
&= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x - Tx - (y-Ty)\|^2 \\
&\le \beta\|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\quad +(1-\beta)\left[\|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\lambda\langle x-Tx, y-Ty\rangle\right] \\
&= \|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\quad +k(1-\beta)\|x-Tx-(y-Ty)\|^2 + 2\lambda(1-\beta)\langle x-Tx, y-Ty\rangle \\
&= \|x-y\|^2 - (1-\beta)(\beta-k)\|x-Tx-(y-Ty)\|^2 \\
&\quad +2\lambda(1-\beta)\langle x-Tx, y-Ty\rangle \\
&\le \|x-y\|^2 + 2\lambda(1-\beta)\langle x-Tx, y-Ty\rangle \\
&= \|x-y\|^2 + 2\lambda(1-\beta)\langle \frac{x-T_\beta x}{1-\beta}, \frac{y-T_\beta y}{1-\beta}\rangle \\
&= \|x-y\|^2 + \frac{2\lambda}{(1-\beta)}\langle x-T_\beta x, y-T_\beta y\rangle. \qquad (3.2)
\end{aligned}$$

It follows from (3.2) that T_β is quasi-nonexpansive. Let $p \in F(T)$. We have

$$\begin{aligned}
\|z_n - p\| &= \|\frac{1}{n}\sum_{k=0}^{n-1} T_\beta^k x_n - p\| \\
&\le \frac{1}{n}\sum_{k=0}^{n-1}\|T_\beta^k x_n - p\| \le \frac{1}{n}\sum_{k=0}^{n-1}\|x_n - p\|| \le \|x_n - p\|. \qquad (3.3)
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)z_n - p\| \\
&= \|\alpha_n u + (1-\alpha_n)z_n - \alpha_n p + \alpha_n p - p\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\|z_n - p\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|. \qquad (3.4)
\end{aligned}$$

By (3.4) and induction, we can conclude that for all $n \in \mathbb{N}$,

$$\|x_n - p\| \le \max\{\|u-p\|, \|x_1 - p\|\}.$$

[6] Thus $\{x_n\}$ and $\{z_n\}$ are bounded. Since $\|T_\beta^n x_n - p\| \le \|x_n - p\|$, we have that $\{T_\beta^n x_n\}$ is also bounded. Observe that since $\{z_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\alpha_n = 0$, then

$$\begin{aligned} \|x_{n+1} - z_n\| &= \|\alpha_n u + (1-\alpha_n) z_n - p\| \\ &= \alpha_n \|u - z_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (3.5)$$

We may assume without loss of generality that exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty} \langle u - P_{F(T)}u, x_n - P_{F(T)}u\rangle = \lim_{j\longrightarrow\infty} \langle u - P_{F(T)}u, x_{n_j} - P_{F(T)}u\rangle,$$

and $x_{n_j} \rightharpoonup w$ as $j \longrightarrow \infty$. Since $\|x_{n+1} - z_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, it follows that $z_{n_j} \rightharpoonup w$ as $j \longrightarrow \infty$. Next, we will show that $w \in F(T)$. Using (3.2) we obtain for all $k = 0, 1, 2, ..., n-1$ and for arbitrary $y \in C$

$$\begin{aligned} \|T_\beta^{k+1} x_n - T_\beta y\|^2 &= \|T_\beta(T_\beta^k x_n) - T_\beta y\|^2 \\ &\leq \|T_\beta^k x_n - y\|^2 + \frac{2\lambda}{1-\beta}\langle T_\beta^k x_n - T_\beta^{k+1} x_n, y - T_\beta y\rangle \\ &= \|T_\beta^k x_n - T_\beta y + T_\beta y - y\|^2 + \frac{2\lambda}{1-\beta}\langle T_\beta^k x_n - T_\beta^{k+1} x_n, y - T_\beta y\rangle \\ &= \|T_\beta^k x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle T_\beta^k x_n - T_\beta y, T_\beta y - y\rangle \\ &\quad \frac{2\lambda}{1-\beta}\langle T_\beta^k x_n - T_\beta^{k+1} x_n, y - T_\beta y\rangle. \end{aligned} \quad (3.6)$$

Summing (3.6) from $k = 0$ to $n-1$ and dividing by n we obtain

$$\begin{aligned} \frac{1}{n}\|T_\beta^n x_n - T_\beta y\|^2 &\leq \frac{1}{n}\|x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_n - T_\beta y, T_\beta y - y\rangle \\ &\quad \frac{2\lambda}{n(1-\beta)}\langle x_n - T_\beta^n x_n, y - T_\beta y\rangle. \end{aligned} \quad (3.7)$$

Since $\{z_n\}$ is bounded, then there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ which converges weakly to $w \in C$. Replacing n by n_j in (3.7) we obtain

$$\begin{aligned} \frac{1}{n_j}\|T_\beta^n x_n - T_\beta y\|^2 &\leq \frac{1}{n_j}\|x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_n - T_\beta y, T_\beta y - y\rangle \\ &\quad \frac{2\lambda}{n_j(1-\beta)}\langle x_n - T_\beta^n x_n, y - T_\beta y\rangle. \end{aligned} \quad (3.8)$$

Since $\{x_n\}$ and $\{T_\beta^n x_n\}$ are bounded, letting $j \longrightarrow \infty$ in (3.8) yields

$$0 \leq \|T_\beta y - y\|^2 + 2\langle w - T_\beta y, T_\beta y - y\rangle. \quad (3.9)$$

Since $y \in C$ was arbitrary, if we set $y = w$ in (3.9) we obtain

$$0 \leq \|T_\beta w - w\|^2 - 2\|T_\beta w - w\|^2,$$

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from which it follows that $w \in F(T_\beta) = F(T)$. Since $P_{F(T)} : H \longrightarrow F(T)$ is the metric projection we have

$$\lim_{j \longrightarrow \infty} \langle u - P_{F(T)}u, x_{n_j} - P_{F(T)}u\rangle = \langle u - P_{F(T)}u, w - P_{F(T)}u\rangle \leq 0.$$

Using Lemma 2.1(ii) and (3.3) we have

$$\begin{aligned} \|x_{n+1} - P_{F(T)}u\|^2 &= \|\alpha_n u + (1-\alpha_n)z_n - P_{F(T)}u\|^2 \\ &= \|\alpha_n u - \alpha_n P_{F(T)}u + (1-\alpha_n)z_n - P_{F(T)}u + \alpha_n P_{F(T)}u\|^2 \\ &= \|\alpha_n u - \alpha_n P_{F(T)}u + (1-\alpha_n)z_n - (1-\alpha_n)P_{F(T)}u\|^2 \\ &= \|\alpha_n(u - P_{F(T)}u) + (1-\alpha_n)(z_n - P_{F(T)}u)\|^2 \\ &\leq (1-\alpha_n)^2\|z_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\ &\leq (1-\alpha_n)^2\|x_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle. \end{aligned} \tag{3.10}$$

Since $\alpha_n \longrightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\longrightarrow\infty}\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \leq 0$, it follows from Lemma 2.3 that $\lim_{n\longrightarrow\infty} \|x_n - P_{F(T)}u\| = 0$.

$$0 \leq \|z_n - P_{F(T)}u\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - P_{F(T)}u\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $\lim_{n\longrightarrow\infty} \|z_n - P_{F(T)}u\| = 0$.

□

Theorem 3.2. Let C be a nonempty closed convex subset of of a real Hilbert space H. Let $T : C \longrightarrow C$ be a k-strictly pseudo λ-hybrid mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ satisfying the conditions:

$$\lim_{n\longrightarrow\infty} \alpha_n = 0 \ \textit{ and } \ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_\beta x_n, \ n \geq 1. \tag{3.11}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of T.

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Proof. It is clear that $F(T_\beta) = F(T) \neq \emptyset$. As in the proof of Theorem 3.1 we have

$$\begin{aligned}
\|T_\beta x - T_\beta y\|^2 &= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\leq \beta\|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\quad +(1-\beta)\left[\|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\lambda\langle x-Tx, y-Ty\rangle\right] \\
&= \|x-y\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\
&\quad +k(1-\beta)\|x-Tx-(y-Ty)\|^2 + 2\lambda(1-\beta)\langle x-Tx, y-Ty\rangle \\
&= \|x-y\|^2 - (1-\beta)(\beta-k)\|x-Tx-(y-Ty)\|^2 \\
&\quad +2\lambda(1-\beta)\langle x-Tx, y-Ty\rangle \\
&= \|x-y\|^2 - \frac{(\beta-k)}{(1-\beta)}\|x-T_\beta x-(y-T_\beta y)\|^2 \\
&\quad +\frac{2\lambda}{(1-\beta)}\langle x-T_\beta x, y-T_\beta y\rangle \\
&\leq \|x-y\|^2 - (\beta-k)\|x-T_\beta x-(y-T_\beta y)\|^2 \\
&\quad +\frac{2\lambda}{(1-\beta)}\langle x-T_\beta x, y-T_\beta y\rangle. \qquad (3.12)
\end{aligned}$$

Thus for all $x \in C$ and for all $p \in F(T) = F(T_\beta)$ we have

$$\|T_\beta x - p\|^2 \leq \|x-p\|^2 - (\beta-k)\|x-T_\beta x\|^2.$$

This implies that T_β is a $quasi-firmly\ type\ nonexpansive$ mapping (see for example [17]). Hence it follows from [17] (see Theorem 3.1 and Remark 1 of [17]) that $\{x_n\}_{n=1}^{\infty}$ convergences strongly to a point $p \in F(T) = F(T_\beta)$. □

By Definition of $k-$ strictly pseudo $\lambda-$ hybrid Mapping, if $k = 0$ and $\lambda = 1$, then T is nonspreading. It follows that we have some corollary.

Corollary 3.1. ([9]) Let C be a nonempty closed convex subset of of a real Hilbert space. Let $T : C \longrightarrow C$ be a nonspreading mapping with a nonempty fixed point set $F(T)$. Let $\beta \in (0,1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ satisfying the conditions:

$$\lim_{n\longrightarrow\infty} \alpha_n = 0 \ \ and \ \ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_\beta x_n, \ n \geq 1. \qquad (3.13)$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of T.[9]

If $\lambda = 1$, then T is $k-$ strictly pseudononspreading.

Corollary 3.2. ([16]) Let C be a nonempty closed convex subset of of a real Hilbert space. Let $T : C \longrightarrow C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1 - \beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ satisfying the conditions:

$$\lim_{n \longrightarrow \infty} \alpha_n = 0 \ \ and \ \ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_\beta x_n, \ n \geq 1. \tag{3.14}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of T.

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