Convergence Theorems for Hemicontractive Mappings

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Abstract

In this paper, we establish strong convergence theorems for the modified Mann-type implicit process of a continuous hemicontractive mappings defined on a noncompact domain in real Hilbert spaces.

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1 Introduction

Let C be a nonempty subset of a real Hilbert space H and let $T : C \to C$ be a mapping. Then

T is said to be pseudocontractive [1, 2] if

$$\| Tx - Ty \|^2 \leqslant \| x - y \|^2 + \| (I - T)x - (I - T)y \|^2 \quad \text{for all } x, y \in C.$$

T is said to be hemicontractive if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ and $\| Tx - p \|^2 \leqslant \| x - p \|^2 + \| x - Tx \|^2$ for all $p \in F(T)$ and $x \in C$. It is easy to see that, if $F(T) \neq \emptyset$, then the concept of hemicontractive mapping is more general than that of pseudocontractive mapping.

Let C be nonempty closed convex subset of a real Hilbert space H. For a mapping $T : C \to C$, the following Mann-type implicit process was introduced by Soltuz [3]:

$$\begin{aligned} x_0 &\in C, \\ x_n &= \alpha_n x_{n-1} + (1 - \alpha_n) T x_n \end{aligned} \tag{1.1}$$

for all $n \geqslant 1$, where $\{\alpha_n\}$ is a real sequence in [0,1].

Definition 1.1. *[4] A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy condition* **A** *if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$\| x - Tx \| \geqslant f(d(x, F(T)))$$

for all $x \in C$, where $d(x, F(T)) = \inf_{z\in F(T)} \| x - z \|$.

Approximating fixed points of Ishikawa [5] (Mann [6]) iterations under a pseudocontractive (or hemicontractive) mapping T has been investigated by several author; see, for example, [7, 13] and others.

In 2014, Kim [8] gave the strong convergence theorems of (1.2) in a real Hilbert space under T is continuous with $F(T) \neq \emptyset$ and T satisfy condition A and T is continuous hemicontractive which $T(C)$ is contains in a compact subset of C.

The purpose of this paper is to introduce and investigatets the following modified Mann implicit iteration process. Let C a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ defined $\{x_n\}$ in C in the following way:

$$\begin{aligned} x_0 &\in C, \\ x_n &= \alpha_n x_{n-1} + (1 - \alpha_n - \beta_n)Tx_n + \beta_n U_n \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$ and $\{U_n\}$ is a bounded sequence in C.

We first prove the strong convergence of (1.1) and (1.2) for a continuous hemicontractive mapping in a real Hilbert space. Next, we give some examples of a hemicontractive mapping which is not a pseudocontractive mapping.

2 Preliminaries

We give some definition, notations and some useful results that will be used in the later section. Throughout this research, we let $\mathbb{R}$ stand for the set of all real numbers and $\mathbb{N}$ for the set of all natural numbers.

Lemma 2.1. [8] Let H be a real Hilbert space. Then the following well know results hold;

1. *(1)* $\| \lambda x + (1-\lambda)y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0,1]$.
2. *(2)* $\| x + y \|^2 \leqslant \| x \|^2 + 2\langle y, x+y \rangle$ *for all* $x, y \in H$.
3. *(3)* $\| x - y \|^2 = \| x \|^2 + \| y \|^2 - 2\langle x, y \rangle$ *for all* $x, y \in H$.

Lemma 2.2. [8] Let C be a nonempty subset of a real Hilbert space H and $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. Then T is hemicontractive if and only if $\langle Tx - p, x - p \rangle \leqslant \| x - p \|^2$ for all $p \in F(T)$ and $x \in C$.

Lemma 2.3. [8] Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a continuous hemicontractive mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed.

Lemma 2.4. [8] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $\Sigma_{n=1}^{\infty} b_n < \infty$ and $a_{n+1} \leqslant a_n + b_n$; for all $n \geqslant 1$. Then $lim_{n\to\infty} a_n$ exists.

3 Main Results

In this section, we prove that if the mapping T is a hemicontractive mapping on C which satisfies some condition, then the sequence $\{x_n\}$ is defined by (1.1) and (1.2) converge strongly to a fixed point of T, respectively.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a continuous hemicontractive mapping with $F(T) \neq \emptyset$, and T satisfies condition* **A**. *Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfies $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. For any $x_0 \in C$, the sequence $\{x_n\}$ is defined by $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n$, for all $n \geqslant 1$. Then $\{x_n\}$ converges strongly to fixed point of T.*

Proof. Let $p \in F(T)$. By using Lemma 2.2, we obtain

$$\| x_n - p \|^2 = \langle x_n - p, x_n - p \rangle = \alpha_n \langle x_{n-1} - p, x_n - p \rangle + (1 - \alpha_n)\langle Tx_n - p, x_n - p \rangle$$
$$\leqslant \alpha_n \| x_{n-1} - p \| \| x_n - p \| + (1 - \alpha_n) \| x_n - p \|^2 .$$

Thus

$$\| x_n - p \| \leqslant \| x_{n-1} - p \| . \tag{3.1}$$

By Lemma 2.4, we have $\lim_{n\to\infty} \|x_n - p\|$ exists. Since T is hemiconractive, we have

$$\|Tx_n - p\|^2 \leqslant \|x_n - p\|^2 + \|x_n - Tx_n\|^2. \tag{3.2}$$

By using sequence $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n$, for all $n \geqslant 1$, we obtain

$$\|x_n - Tx_n\|^2 = \alpha_n^2 \|x_{n-1} - Tx_n\|^2. \tag{3.3}$$

By using Lemma 2.1, (3.2), we have

$$\begin{aligned} \|x_n - p\|^2 &= \|\alpha_n x_{n-1} + (1 - \alpha_n)Tx_n - p\|^2 \\ &= \|\alpha_n (x_{n-1} - p) + (1 - \alpha_n)(Tx_n - p)\|^2 \\ &\leqslant \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\ &\quad + (1 - \alpha_n)\|x_n - Tx_n\|^2 - \alpha_n(1 - \alpha_n)\|x_{n-1} - Tx_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\|x_{n-1}-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \\
&\quad +(1-\alpha_n)\alpha_n^2\|x_{n-1}-Tx_n\|^2-\alpha_n(1-\alpha_n)\|x_{n-1}-Tx_n\|^2 \\
&= \alpha_n\|x_{n-1}-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\alpha_n(1-\alpha_n)^2\|x_{n-1}-Tx_n\|^2.
\end{aligned}
$$

It follows that

$$(1-\alpha_n)^2\|x_{n-1}-Tx_n\|^2 \leqslant \|x_{n-1}-p\|^2-\|x_n-p\|^2.$$

And from the condition $\{\alpha_n\}\subset[\delta,1-\delta]$, we conclude that the inequality

$$\delta^2\|x_{n-1}-Tx_n\|^2 \leqslant \|x_{n-1}-p\|^2-\|x_n-p\|^2.$$

Since, $\lim_{n\to\infty}\|x_n-p\|$ exists, we have

$$\lim_{n\to\infty}(\|x_{n-1}-p|^2-\|x_n-p\|^2).$$

This implies that

$$\lim_{n\to\infty}\|x_{n-1}-Tx_n\|=0. \tag{3.4}$$

From (3.3) and (3.4), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.5}$$

By using Condition A, there exists a nondecreasing mapping $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$f(d(x_n,F(T)))\leqslant\|Tx_n-x_n\|. \tag{3.6}$$

for all $n\geq 1$.

We claim that $\lim_{n\to\infty}d(x_n,F(T))=0$. In fact, assume $\lim_{n\to\infty}d(x_n,F(T))=k>0$. Then we can choose $n_0\in\mathbb{N}$ such that $0<\frac{k}{2}<d(x_n,F(T))$ for all $n\geq n_0$. By using a mapping f and (3.4), we obtain

$$0 < f(\frac{k}{2}) \leqslant f(d(x_n, F(T))) \leqslant \|Tx_n - x_n\| \to 0$$

as $n \to \infty$. This a contradiction. So, we obtain $k = 0$. For any $\epsilon > 0$ be given, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \frac{\epsilon}{2}, \tag{3.7}$$

for all $n \geqslant n_0$. Then for all $n, m \geqslant n_0$ and $v \in F(T)$, we obtain

$$\|x_n - x_m\| \leqslant \|x_n - v\| + \|x_m - v\| \leqslant 2\|x_{n_0} - v\|. \tag{3.8}$$

Taking the infimum over all $v \in F(T)$ on both sides and by (3.8), we obtain

$$\|x_n - x_m\| \leqslant 2[d(x_{n_0}, F(T))] < \epsilon, \tag{3.9}$$

for all $n, m \geqslant n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Then $\{x_n\}$ is a convergent sequence, we may assume that $\lim_{n\to\infty} x_n = w$. Then $d(w, F(T)) = 0$. By Lemma (2.3), $F(T)$ is closed and thus we obtain $w \in F(T)$. Hence $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a continuous hemicontractive mapping with $F(T) \neq \emptyset$, and T satisfies condition **A**. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences with $\alpha < \alpha_n, \beta_n \leq 1$ for some $\alpha \in (0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\beta_n \leq \alpha_n^2$. Then a sequence $\{x_n\}$ is defined by (1.2) converges strongly to fixed point of T.*

Proof. Let $p \in F(T)$. By using Lemma 2.2, we obtain

$$\begin{aligned} \|x_n - p\|^2 &= \langle x_n - p, x_n - p\rangle \\ &= \langle(\alpha_n x_{n-1} + (1 - \alpha_n - \beta_n)Tx_n + \beta_n U_n) - p, x_n - p\rangle \end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\langle x_{n-1} - p, x_n - p\rangle + (1-\alpha_n-\beta_n)\langle Tx_n - p, x_n - p\rangle \\
&+ \beta_n\langle U_n - p, x_n - p\rangle \\
&= \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1-\alpha_n-\beta_n)\|Tx_n - p\|\|x_n - p\| \\
&+ \beta_n\|U_n - p\|\|x_n - p\| \\
&= \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1-\alpha_n-\beta_n)\|x_n - p\|^2 \\
&+ \beta_n\|U_n - p\|\|x_n - p\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_n - p\| &\leqslant \alpha_n\|x_{n-1} - p\| + (1-\alpha_n-\beta_n)\|x_n - p\| + \beta_n\|U_n - p\| \\
&\leqslant \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\|x_n - p\| + \beta_n\|U_n - p\|.
\end{aligned}
$$

Since, $\alpha < \alpha_n, \beta_n \leq 1$ for some $\alpha \in (0,1]$ and $\beta_n \leq \alpha_n^2$, we obtain that

$$
\begin{aligned}
\alpha_n\|x_n - p\| &\leqslant \alpha_n\|x_{n-1} - p\| + \beta_n\|U_n - p\| \\
&\leqslant \alpha_n\|x_{n-1} - p\| + \alpha_n^2\|U_n - p\|.
\end{aligned}
$$

Hence

$$
\|x_n - p\| \leqslant \|x_{n-1} - p\| + \alpha_n\|U_n - p\|.
$$

Since, $\sum_{n=1}^{\infty}\alpha_n < \infty$ and $\{\|U_n - p\|\}$ is bounded, then $\sum_{n=1}^{\infty}\alpha_n\|U_n - p\| < \infty$. By Lemma 2.4, we have $\lim_{n\to\infty}\|x_n - p\|$ exists. Since T is hemicontractive, we have

$$
\begin{aligned}
\|x_n - Tx_n\| &= \|\alpha_n x_{n-1} + (1-\alpha_n-\beta_n)Tx_n + \beta_n U_n - Tx_n\| \\
&= \|\alpha_n x_{n-1} + Tx_n - \alpha_n Tx_n - \beta_n Tx_n + \beta_n U_n - Tx_n\| \\
&= \|\alpha_n x_{n-1} - \alpha_n Tx_n + \beta_n U_n - \beta_n Tx_n\| \\
&= \|\alpha_n(x_{n-1} - Tx_n) + \beta_n(U_n - Tx_n)\| \\
&\leqslant \alpha_n\|x_{n-1} - Tx_n\| + \beta_n\|U_n - Tx_n\|.
\end{aligned}
$$

Since, $\alpha < \alpha_n, \beta_n \leq 1$ we get $\alpha_n^2 \leq \alpha_n$ then $\beta_n \leqslant \alpha_n^2 \leq \alpha_n$ we have

$$\|x_n - Tx_n\| \leqslant \alpha_n\|x_{n-1} - Tx_n\| + \alpha_n\|U_n - Tx_n\|. \tag{3.10}$$

Since $\{x_n\}$ is a bounded, T is continuous and $\Sigma_{n=1}^{\infty}\alpha_n < \infty$ then

$$\lim_{n\to\infty} \alpha_n\|x_{n-1} - Tx_n\| = 0 = \lim_{n\to\infty} \alpha_n\|U_n - Tx_n\|. \tag{3.11}$$

From (3.10) and (3.11), it implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.12}$$

By using Condition A, there exists a nondecreasing mapping $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$f(d(x_n, F(T))) \leqslant \|Tx_n - x_n\|. \tag{3.13}$$

for all $n \geq 1$.

We claim that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. In fact, assume $\lim_{n\to\infty} d(x_n, F(T)) = k > 0$. Then we can choose $n_0 \in \mathbb{N}$ such that $0 < \frac{k}{2} < d(x_n, F(T))$ for all $n \geq n_0$. By using a mapping f and (3.4), we obtain

$$0 < f(\frac{k}{2}) \leqslant f(d(x_n, F(T))) \leqslant \|Tx_n - x_n\| \to 0 \text{ as } n \to \infty.$$

This a contradiction. So, we obtain $k = 0$. For any $\epsilon > 0$ be given, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \frac{\epsilon}{2}, \tag{3.14}$$

for all $n \geqslant n_0$. Then for all $n, m \geqslant n_0$ and $v \in F(T)$, we obtain

$$\|x_n - x_m\| \leqslant \|x_n - v\| + \|x_m - v\| \leqslant 2\|x_{n_0} - v\|. \tag{3.15}$$

Taking the infimum over all $v \in F(T)$ on both sides and by (3.15), we obtain

$$\|x_n - x_m\| \leqslant 2[d(x_{n_0}, F(T))] < \epsilon, \tag{3.16}$$

for all $n, m \geqslant n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Then $\{x_n\}$ is a convergent sequence, we may assume that $\lim_{n\to\infty} x_n = w$. Then $d(w, F(T)) = 0$. By Lemma (2.3), $F(T)$ is closed and thus we obtain $w \in F(T)$. Hence $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.3. *Let C be nonempty closed convex subset of a real Hilbert space, and $T : C \to C$ be a continuous hemicontractive mapping, and let $T(C)$ be contained in a compact subset of C. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences in (0,1] satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\beta_n \leqslant \alpha_n^2$. For any $x_0 \in C$, the sequence $\{x_n\}$ is defined by (1.2). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Mazur's theorem [14], $W := \overline{co}(\{x_0\} \bigcup T(C))$ is a compact subset of C containing $\{x_n\}$ which is invariant under T. So, without loss of generality, we may assume that C is compact and $\{x_n\}$ is well-defined. The existence of a fixed point of T follows from Schauder's fixed theorem [15]. Since C is compact and by (3.12) in the proof of Theorem 3.2, there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ and a point $z \in C$. By using (3.12) in the proof Theorem 3.2 and the continuity of T, we obtain $z \in F(T)$. Hence, by (**??**) in the proof of Theorem 3.2, we obtain $\lim_{n\to\infty} \| x_n - z \| = 0$. □

If $\beta_n \equiv 0$, then we have the following result :

Corollary 3.4. *Let C be nonempty closed convex subset of a real Hilbert space, and $T : C \to C$ be a continuous hemicontractive mapping, and let $T(C)$ be contained in a compact subset of C. Let $\{\alpha_n\}$ be a real sequence in (0,1] satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$. For any $x_0 \in C$, the sequence $\{x_n\}$ is defined by (1.1). Then $\{x_n\}$ converges strongly to a fixed point of T.*

We give an example of a hemicontractive mapping which is not a pseudocontractive mapping.

Example 3.5. *Let $H = \mathbb{R}$ and $C = [-2\pi, 2\pi]$ and let $T : C \to C$ be defined by*

$$Tx = \frac{x}{2}\sin x \quad \textit{for each} \quad x \in C.$$

Obviously, $F(T) = \{0\}$ and T is hemicontractive mapping, that is, if $x \in C$ and $p = 0$, then

$$| Tx - p |^2 = | Tx - 0 |^2 = | \frac{x}{2}\sin x |^2 \leqslant | x |^2 = | x - p |^2 .$$

Thus $| Tx - p |^2 \leqslant | x - p |^2 + | x - Tx |^2$.

But it is not a pseudocontractive mapping. In fact, if we take $x = 2\pi$ and $y = \frac{3\pi}{2}$, then

$$| Tx - Ty |^2 = | \pi \sin 2\pi - \frac{3\pi}{4}\sin\frac{3\pi}{2} |^2 = | \frac{3\pi}{4} |^2 = \frac{9\pi^2}{16},$$

where, $| x - y |^2 = \frac{\pi^2}{4}$ and $| (x - Tx) - (y - Ty) |^2 = | \frac{\pi}{2} - \frac{3\pi}{4} = | \frac{\pi}{4} | = \frac{\pi}{16}$. □

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