สมการไดโอแฟนไทน์คิวบิค 3 ตัวแปร $11(x^2+y^2)-21xy+x+y+1=68z^3$

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บทคัดย่อ

สมการไดโอแฟนไทน์คิวบิค 3 ตัวแปรไม่เอกพันธ์กำหนดโดย $11(x^2+y^2)-21xy+x+y+1=68z^3$ ถูกพิจารณาเพื่อหารูปแบบสำหรับผลเฉลยที่ไม่เป็นศูนย์ มีการนำเสนอคุณสมบัติที่น่าสนใจบางประการระหว่างผลเฉลยและจำนวนเต็มพิเศษ

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On Ternary Cubic Diophantine Equation $11(x^2+y^2)-21xy+x+y+1=68z^3$

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Abstract

The non-homogeneous ternary cubic diophantine equation respresented by $11(x^2+y^2)-21xy+x+y+1=68z^3$ is considered for its patterns of non-zero distinct integral solutions. A few fascinaling properties among the solutions and special integer are presented.

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1 Introduction

The number theory is the king of Mathematics. In particular the diophantine equations have a blend of attracted interesting problem. In 2014, S. Vidhyalakshmi, T.R. UshaRani and M.A.Gopalan [13] has considered the ternary cubic equation $5(x^2 + y^2) - 9xy + x + y + 1 = 35z^3$ and determining its non-zero distinct integral solutions. Employing the linear transformations $x = u + v, y = u - v(u \neq v \neq 0)$, and employing the method of factorization in complex conjugates, different patterns of integral solutions to the ternary cubic equation under consideration are obtained.

In 2015, V. Manju Somanath, M.A. Sangeetha, Gopalan and M. Bhuvaneshwari [14] are stdied on ternary cubic diophantine equation

$$3(x^2 + y^2) - 2xy + 4(x + y) + 4 = 51z^3. \tag{1.1}$$

In 2016, R. Anbuselvi and S. A. Shanmugavadivu [11] are considered for determining $5(x^2+y^2)-9xy+x+y+1 = 28z^3$ non-zero distinct integral solutions. R.Anbuselvi and K.Kannaki [8] are interested equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$ representing non-homogeneous cubic equation with three unknowns for determining its infinitely many non-zero integral points. R. Anbuselvi and R.Nandhini [10] studied seven different methods of the non-zero non-negative solutions of homogeneous Diophantine equation $5(x^2 + y^2) - 9xy = 23z^3$. P. Jayakumar and J. Meena [7] are observed another interesting four different methods of the non-zero non-negative solutions the non - homogeneous cubic diophantine equation $x^2 + y^2 - xy = 39z^3$. Next, G.Janaki and C.Saranya [2] analyed for the diophantine equation $3(x^2+y^2)-4xy+2(x+y+1) = 972z^3$ on non-zero integral solutions.

In 2018, R. Anbuselvi and R. Nandhini [9] studied observations on the ternary cubic diophantine equation $x^2 + y^2 - xy = 52z^3$ and its infinitely many non-zero distinct integral solutions.

In 2022, S. Mallika, S. Vidhyalakshmi, K. Hema and M. A. Gopalan [12] are obtained different sets of non-zero distinct integral solutions of ternary non-homogeneous cubic diophantine equation $x^2 + xy + y^2 = (m^2 + 3n^2)z^3$.

In this communication, we are observed another interesting four different methods of the non-zero non-negative solutions the non-homogeneous cubic diophantine equations $11(x^2+y^2)-21xy+x+y+1=68z^3$. Further, some elegant properties among the special numbers and the solutions are observed. In this work, we will denote the notations is the following.

- $t_{m,n}$ = Polygonal number of rank n with side m.
- SO_n= Stella octangula number of rank n.
- CP_n^6= Centered hexagonal pyramidal number of rank n.

2 Main Results

The ternary cubic diophantine equation to be solved to be for its non-zero distinct integral solutions is given by

$$11(x^2+y^2)-21xy+x+y+1=68z^3. \tag{2.1}$$

Set

$$x=u+v \text{ and } y=u-v. \tag{2.2}$$

Then (2.1), it leads to

$$(u+1)^2+43v^2=68z^3. \tag{2.3}$$

Pattern-I

Assume

$$z=a^2+43b^2. \tag{2.4}$$

Write 68 as

$$68=(5+i\sqrt{43})(5-i\sqrt{43}). \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3) and employing the factorization its is expressed as

$$((u+1)+i\sqrt{43}v)((u+1)-i\sqrt{43}v)=(5+i\sqrt{43})(5-i\sqrt{43})(a+i\sqrt{43}b)^3(a-i\sqrt{43}b)^3. \tag{2.6}$$

By combination, equation (2.6) which is corresponding to the system of equation

$$(u+1)+i\sqrt{43}v=(5+i\sqrt{43})(a+i\sqrt{43}b)^3 \tag{2.7}$$

and

$$(u+1)-i\sqrt{43}v=(5-i\sqrt{43})(a-i\sqrt{43}b)^3. \tag{2.8}$$

Comparing the positive and negative parts either in (2.7) or (2.8), we get

$$u = 5a^3-129a^2b-645ab^2+1849b^3-1,$$
$$v = a^3+15a^2b-129ab^2-215b^3.$$

In sight of (2.2), the non - zero different integral solution of (2.1), are

$$x = x(a,b) = 6a^3-114a^2b-774ab^2+1634b^3-1$$
$$y = y(a,b) = 4a^3-144a^2b-516ab^2+2064b^3-1,$$
$$z = z(a,b) = a^2+43b^2.$$

The equation of $x(a,b), y(a,b)$ and $z(a,b)$ represent non-zero distinct integral solution of (2.1) on to parameters.

Observation

(i) $11z(a,a)$ is perfect square.

(ii) $484[y(a,a)+1]$ is cubical integer.

(iii) $143[x(-a,a)+1]$ is perfect square.

(iv) $z(a,a)-CP_n^6 \equiv 0\ (mod\ 43)$.

(v) $2y(1,1)-3x(1,1)-13z(1,1)+11t_{6,1}=0$.

(vi) $y(a,1)-x(a,1)-11So_a+9t_{6,a} \equiv 2\ (mod\ 4)$.

Pattern-II

By combination equation (2.6) can be written as ;

$$(u+1)+i\sqrt{43}v = (5+i\sqrt{43})(a-i\sqrt{43}b)^3.$$
$$(u+1)-i\sqrt{43}v = (5-i\sqrt{43})(a+i\sqrt{43}b)^3.$$

We have

$$u=5a^3+129a^2b-645ab^2-1849b^3-1$$

and

$$v = 15a^2b - 215b^3 - a^3 + 129ab^2$$

which implies the non-zero different integral solution of (2.1), are

$$x = x(a,b) = 4a^3 + 144a^2b - 516ab^2 - 2064b^3 - 1$$

$$y = y(a,b) = 6a^3 + 114a^2b - 774ab^2 - 1634b^3 - 1$$

$$z = z(a,b) = a^2 + 43b^2.$$

The equation of $x(a,b), y(a,b)$ and $z(a,b)$ represent non-zero distinct integral solution of pattern II on to parameters.

Observation

(i) $y(a,a) + 1 \equiv 0 \ (mod\ 143)$.

(ii) $x(a,a) + 1 \equiv 0 \ (mod\ 19)$.

(iii) $x(a,2) - y(a,2) + 2SO_a \equiv 4 \ (mod\ 12)$.

(iv) $x(9,2) + 27z(9,2) + 19t_{6,2} \equiv 0 \ (mod\ 909)$.

Pattern-III

We have equation (2.3) can also be written as

$$(u+1)^2 + 43v^2 = 68z^3 * 1 \tag{2.9}$$

Put 1 as,

$$1 = \frac{(21 + i\sqrt{43})(21 - i\sqrt{43})}{484} \tag{2.10}$$

Using the method of factorization as in pattern I comparing the positive parts, we get

$$x = \frac{1}{11}(44a^3 - 1584a^2b - 5676ab^2 + 22704b^3) - 1$$

$$y = \frac{1}{11}(18a^3 - 1770a^2b - 2322ab^2 + 25370b^3) - 1$$

$$z = a^2 + 43b^2.$$

Put $a = 11A$ and $b = 11B$, we have

$$x = 5324A^3 - 191664A^2B - 686796AB^2 + 2747184B^3 - 1$$

$$y = 2178A^3 - 214170A^2B - 280962AB^2 + 3069770B^3 - 1$$

$$z = 121A^2 + 5203B^2.$$

The equation of $x(A,B), y(A,B)$ and $z(A,B)$ represent non-zero distinct integral solutions of pattern III on to parameters.

Observation

(i) $y(A,1) - x(A,1) + z(A,1) \equiv 0 \ (mod\ 11)$.

(ii) $y(A,1) - 6z(A,1) + 27t_{4,A} \equiv 1 \ (mod\ 3)$.

(iii) If A is odd, then $y(A,1) - 7z(A,1) + 132t_{6,A} \equiv 43 \ (mod\ 45)$, and

if A is even, then $y(A,1) - 7z(A,1) + 132t_{6,A} \equiv 32 \ (mod\ 45)$.

(iv) $x(A,2) + 23z(A,2) - 11SO_A \equiv 10 \ (mod\ 11)$.

Pattern-IV

We have equation (2.3) can also be written as

$$(u+1)^2 + 43v^2 = 68z^3 * 1. \tag{2.11}$$

Put 1 as,

$$1 = \frac{(17+3i\sqrt{43})(17-3i\sqrt{43})}{676}.$$

Using the method of factorization as in pattern I comparing the positive parts, we get

$$u = \frac{1}{13}(-22a^3 - 2064a^2b + 2838ab^2 + 29584b^3) - 1,$$
$$v = \frac{1}{13}(16a^3 - 66a^2b - 2064ab^2 + 946b^3).$$

Take $a = 13A$ and $b = 13B$, we get

$$u = -3718A^3 - 348816A^2B + 4796622AB^2 + 4999696B^3 - 1,$$
$$v = 2704A^3 - 11154A^2B - 348816AB^2 + 159874B^3.$$

Our interest is to obtain the integer solutions, so that the values of x and y are integers for suitable choices of the parameters A and B.

$$x = -1014A^3 - 359970A^2B + 130806AB^2 + 5159570B^3 - 1$$
$$y = -6422A^3 - 337662A^2B + 828438AB^2 + 4839822B^3 - 1$$
$$z = 169A^2 + 7267B^2.$$

Conclusion In this research, we have made an attempt to obtain all integer solutions to tenary cubic equation $11(x^2+y^2) - 21xy + x + y + 1 = 68z^3$. One may search for other patterns of solutions and their corresponding properties.

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