# กรุปอินเวอร์สของเมทริกซ์ขนาด $3\times3$ เหนือริง

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## The Group Inverse of $3\times3$ Matrices Over a Ring.

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**Abstract**

In this paper, we study conditions for the existence of the group inverse of the $3\times3$ matrix $N = \begin{bmatrix} e & h_1 & h_2 \\ v_1 & a & c \\ v_2 & b & d \end{bmatrix}$ over an arbitrary ring $R$ with unity $1$, when $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is the submatrix of $N$ has the group inverse in $R^{2\times2}$.

**Keywords**:  von Neumann regularity, $\{1,2\}$ -inverse, Group inverse, Matrix over a ring.

## Introduction

Let $\mathbb{C}$ and $\mathbb{R}$ be the field of complex numbers and real numbers respectively. For a positive integers $m,n$, let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ matrices over $\mathbb{C}$. The set of all complex vectors, or $n\times1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$. We denote the identity and the zero matrix in $\mathbb{C}^{n\times n}$ by $I_n$ and $O_n$, respectively. Note that $A^*$ stands for $(\bar{A})^{\mathrm{T}}$.

For a given $A\in\mathbb{C}^{m\times n}$, the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying

$$AXA = A, \tag{1}$$
$$XAX = X, \tag{2}$$
$$(AX)^* = AX, \tag{3}$$
$$(XA)^* = XA, \tag{4}$$

is called the *Moore-Penrose inverse* of $A$ and is denoted by $A^{\dagger}$ (see Ben-Israel, & Greville, 2003).

We also consider the following equations which are applicable to square matrices

$$AX = XA. \tag{5}$$

For a given $A\in\mathbb{C}^{m\times n}$, the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying (1), (2), and (5) is called the *group inverse* of $A$ and denoted by $A^{\#}$.

Unlike the Moore-Penrose inverse, which always exists, the group inverse need not exist for all square matrices. A well known necessary and sufficient condition for the existence of $A^{\#}$ is that $\mathrm{rank}(A) = \mathrm{rank}(A^2)$. If $A$ is nonsingular, then $A^{\#} = A^{-1} = A^{\dagger}$.

The group inverse has applications in singular differential and difference equations, Markov chains and iterative methods. Heinig, 1997, pp. 321-342 investigated the group inverse of Sylvester transformation. Wei, & Diao, 2005, pp. 109-123 studied the representation of the group inverse of a real singular Toeplitz matrix which arises in scientific computing and engineering. Catral, Olesky, &

Driessche, 2008, pp. 219-233 studied the existence of $A^{\#}$ (see Cao, Ge, Wang, & Zhang, 2013).

If $A$ and $B$ are square invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. However, for a generalized inverse this need not be true. Rajesh Kannan & Bapat, 2014 (Theorem 2.2) asserted that for $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$, $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if $BB^{*}A^{\dagger}A$ and $A^{*}ABB^{\dagger}$ are Hermitian.

### Preliminaries and Auxiliary Results

We shall be now concerned with generalized inverses that satisfy some, but not all, of the four Penrose equations.

**Definition** 2.1. [Ben-Israel & Greville, 2003, p. 40]. For any $A \in \mathbb{C}^{m \times n}$, let $A\{i, j, \ldots, k\}$ denote the set of the matrices $X \in \mathbb{C}^{n \times m}$ which satisfies equations $(i)$, $(j)$, ..., $(k)$ from among equations (1) - (4). A matrix $X \in A\{i, j, \ldots, k\}$ is called an $\{i, j, \ldots, k\}$- inverse of $A$, and also denoted by $A^{(i, j, \ldots, k)}$.

The examples are $\{1\}$- inverse (inner inverse), $\{1, 2\}$- inverse (reflexive inner inverse), $\{1, 3\}$- inverse (least squares inner inverse), $\{1, 4\}$- inverse (minimum norm inner inverse), $\{1, 2, 3\}$- inverse, $\{1, 2, 4\}$- inverse and $\{1, 2, 3, 4\}$- inverse, the last being the Moore-Penrose inverse of $A$.

Let $R$ be a ring with unity $1$. An element $a \in R$ is said to be von Neumann regular (regular) if there exists an element $a^{-}$ of $R$ such that $aa^{-}a = a$. In this case, $a^{-}$ is called a $\{1\}$- inverse of $a$. An element $a^{+}$ of $R$ is a $\{1, 2\}$- inverse of $a$, which is given by $a^{+} = a^{-}aa^{-}$ (see [4] for example). Let $R$ be a ring, not necessarily commutative.

Recall that an element $a \in R$ is said to be a unit if it has an inverse, if there is an element $a^{-1} \in R$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ (see for example Bhaskara Rao, 2002 (p.16). We denote an arbitrary $\{1\}$- inverse of $A$ by $A^{-}$ and $\{1, 2\}$- inverse of $A$, which is given

by $A^{+} = A^{-}AA^{-}$. In the next section we will use the following result on regularity.

**Lemma** 2.2. Let $A \in R^{3 \times 3}$ be regular, $B \in R^{3 \times 3}$, be such that there exists $A^{+}$ such that $A^{+}B = B^{+}A = O_{3}$. If $(A + B)$ is regular then $B$ is regular and $B^{-} = (A + B)^{-}$ is a $\{1\}$- inverse of $B$, for any $(A + B)^{-}$.

**Proof:** Since $A^{+}B = B^{+}A = O_{3}$, then

$B = (I_{2} - AA^{+})(A + B) = (A + B)(I_{2} - A^{+}A)$. Hence
$B(A + B)^{-}B = (I_{2} - AA^{+})(A + B)(A + B)^{-}(A + B)(I_{2} - A^{+}A)$
$= (I_{2} - AA^{+})(A + B)(I_{2} - A^{+}A) = B$.

Therefore, $B$ is regular and $(A + B)^{-}$ is an $\{1\}$- inverse of $B$.                    $\square$

Recently, Patricio, & Hartwig, 2010 characterize the existence of the group inverse of a two by two matrix with zero $(2, 2)$ entry, over an arbitrary ring. Castro-Gonzalez, Robles, & Velez-Cerrada, 2013 gave the conditions for the existence of the $2 \times 2$ the matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ has group inverse in $R^{2 \times 2}$ in a ring with unity $1$, and derived a representation of the group inverse of $M$ in the case when either the entry $a$ or $d$ has a group inverse in $R$. Cao, et al., 2013 studied the group inverse of $2 \times 2$ block matrices $M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ over rings $R$ with unity $1$, where $CA = C$, $AB = B$, and the group inverse of $D - CB$ exists. In this paper, we study the group inverse of $3 \times 3$ matrices

$$N = \begin{bmatrix} e & h_{1} & h_{2} \\ v_{1} & a & c \\ v_{2} & b & d \end{bmatrix} =: \begin{bmatrix} e & h^{T} \\ v & M \end{bmatrix} \tag{2.1}$$

over an arbitrary ring $R$ with unity $1$, under the condition the $2 \times 2$ submatrix $M$ has group inverse.

Now we assume that $M \in R^{2 \times 2}$ is regular and that $M^{+}$ is a fixed but arbitrary $\{1, 2\}$- inverse of $M$. Let us introduce the notation

$$E = I_{2} - MM^{+}, \quad F = I_{2} - M^{+}M, \quad s = e - h^{T}M^{+}v. \tag{2.2}$$

We note that $FF = F$ and $EE = E$, since

$$EE = (I_2 - MM^+)(I_2 - MM^+) \quad = (I_2 - MM^+) - MM^+(I_2 - MM^+)$$
$$= (I_2 - MM^+) - (MM^+ - MM^+MM^+)$$
$$= I_2 - MM^+ - MM^+ + (MM^+M)M^+$$
$$= I_2 - MM^+ - MM^+ + MM^+$$
$$= I_2 - MM^+ = E,$$

and

$$FF = (I_2 - M^+M)(I_2 - M^+M) \quad = (I_2 - M^+M) - M^+M(I_2 - M^+M)$$
$$= I_2 - M^+M - (M^+M - M^+MM^+M)$$
$$= I_2 - M^+M - M^+M + (M^+MM^+)M$$
$$= I_2 - M^+M - M^+M + M^+M$$
$$= I_2 - M^+M = F.$$

We can decompose the matrix $N \in R^{3 \times 3}$ as follows.

**Lemma** 2.3. *The matrix $N$ in (2.1) can be factored into*

$$N = \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ M^+v & I_2 \end{bmatrix} =: XZY , \quad (2.3)$$

*where*

$$X = \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix}, \; Z = \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 0^T \\ M^+v & I_2 \end{bmatrix}.$$

**Proof:** Consider

$$XZY = \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ M^+v & I_2 \end{bmatrix}$$
$$= \begin{bmatrix} s + h^T M^+ Ev + h^T FM^+v + h^T M^+ MM^+v & h^T F + h^T M^+ M \\ Ev + MM^+v & M \end{bmatrix}$$
$$= \begin{bmatrix} (e - h^T M^+v) + h^T M^+(I_2 - MM^+)v + h^T(I_2 - M^+M)M^+v + h^T M^+v & h^T F + h^T M^+ M \\ Ev + MM^+v & M \end{bmatrix}$$
$$= \begin{bmatrix} e & h^T \\ v & M \end{bmatrix} = N.$$

□

We have a useful characterization the $N^{\#}$ as in [4, Lemma 1.2].

**Proof:** We must show that,

$$EM = MF = O_2, \; FM^+ = M^+E = O_2, \; yh^T F = (h^T F)^+ y = O_2,$$

where $O_2$ is the zero square matrix of order 2.

Firstly, from (2.2), $E = I_2 - MM^+$, $F = I_2 - M^+M$, we have

$$EM = (I_2 - MM^+)M = M - MM^+M = O_2 = M - MM^+M = M(I_2 - M^+M) = MF,$$
$$FM^+ = (I_2 - M^+M)M^+ = M^+ - M^+MM^+ = O_2 = M^+ - M^+MM^+ = M^+(I_2 - MM^+) = M^+E,$$
$$yh^T F = (1 - h^T F(h^T F)^+)h^T F = h^T F - h^T F(h^T F)^+ h^T F = h^T F - h^T F = O_2,$$
$$(h^T F)^+ y = (h^T F)^+(1 - h^T F(h^T F)^+) = (h^T F)^+ - (h^T F)^+ h^T F(h^T F)^+ = (h^T F)^+ - (h^T F)^+ = O_2.$$

Now, consider

**Proposition** 2.4. Let $X, Y, Z \in R^{3 \times 3}$. If $N = XZY$ where $X$ and $Y$ are units and $Z$ is regular, then the group inverse of $N$ exists if and only if $T = ZYX + I_2 - ZZ^-$ is a unit of $R^{3 \times 3}$, independent of the choice of $Z^-$. Equivalently, $S = YXZ + I_2 - Z^-Z$ is a unit, in which case

$$N^{\#} = XT^{-2}ZY = XZS^{-2}Y.$$

**The group inverse of a $3 \times 3$ matrix over ring with unity** 1.

In the notation (2.2), we assume both $Ev$ and $h^T F$ to be regular elements in $R^{2 \times 1}$ and $R^{1 \times 2}$ respectively. Set

$$x = 1 - (Ev)^+ Ev, \; y = 1 - h^T F(h^T F)^+, \; w = ysx, \quad (3.1)$$

for fixed but arbitrary $(Ev)^+$ and $(h^T F)^+$, and $s$ defined in (2.2). By direct computation, we see that $xx = x$ and $yy = y$.

The von Neumann regularity of the matrix $Z \in R^{3 \times 3}$ defined as in (2.3) is characterized in terms of the regularity of $w$ as an element of the ring $R$ in our next lemma. A representation for a {1}-inverse of $A$ when it exists will prove extremely useful in the solution to our problem.

**Lemma** 3.1. *Let $E, F, s, x, y$ and $w$ be as in (2.2) and (3.1). We have that $Z = \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix}$ is regular in $R^{3 \times 3}$ if and only if $w$ is regular in $R$.*

$$\begin{bmatrix} 1 & -ys(Ev)^+E \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} s & h^TF \\ Ev & M \end{bmatrix}\begin{bmatrix} 1 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix} = \begin{bmatrix} s-ys(Ev)^+EEv & h^TF-ys(Ev)^+EM \\ Ev & M \end{bmatrix}\begin{bmatrix} 1 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} s-ys(Ev)^+Ev-h^TF(h^TF)^+s+ys(Ev)^+0F(h^TF)^+s & h^TF-ys(Ev)^+0 \\ Ev-0(h^TF)^+s & M \end{bmatrix}$$

$$= \begin{bmatrix} s-(1-h^TF(h^TF)^+)s(Ev)^+Ev-h^TF(h^TF)^+s & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} s-s(Ev)^+Ev+h^TF(h^TF)^+s(Ev)^+Ev-h^TF(h^TF)^+s & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} s\left[1-(Ev)^+Ev\right]-h^TF(h^TF)^+s\left[1-(Ev)^+Ev\right] & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} sx-h^TF(h^TF)^+sx & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} \left[1-h^TF(h^TF)^+\right]sx & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} ysx & h^TF \\ Ev & M \end{bmatrix}.$$

Let us denote

$$PZQ := \begin{bmatrix} 1 & -ys(Ev)^+E \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} s & h^TF \\ Ev & M \end{bmatrix}\begin{bmatrix} 1 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} ysx & h^TF \\ Ev & M \end{bmatrix} = \begin{bmatrix} w & h^TF \\ Ev & M \end{bmatrix} =: \tilde{Z}. \tag{3.2}$$

From above $PZQ = \tilde{Z}$, consider $P$ and $Q$ are nonsingular matrices. We have

$$\tilde{Z} = PZQ = P(ZZ^-Z)Q = PZ(QQ^{-1})Z^{-1}(P^{-1}P)ZQ = PZQ(Q^{-1}Z^{-1}P^{-1})PZQ = \tilde{Z}(Q^{-1}Z^{-1}P^{-1})\tilde{Z}.$$

Therefore $Z$ is regular if and only if $\tilde{Z}$ is regular.

Using $w = ysx$ from (3.1), we can write

$$\tilde{Z} = \begin{bmatrix} 0 & h^TF \\ Ev & M \end{bmatrix} + \begin{bmatrix} w & 0^T \\ 0 & O_2 \end{bmatrix} = H + W.$$

Next, we must show that $\tilde{Z}$ is regular if and only if $w$ is regular. We have that $H$ is regular, which

$$H^- = \begin{bmatrix} 0 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix}\begin{bmatrix} xw^-y & (Ev)^+E \\ -F(h^TF)^+ & M^+ \end{bmatrix}\begin{bmatrix} 0 & -ys(Ev)^+E \\ 0 & I_2 \end{bmatrix}$$

is a {1}-inverse of $Z$, and

$$H^+ = \begin{bmatrix} 0 & (EM)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix}$$

is a {1, 2}-inverse of $H$ such that

$WH^+ = H^+W = O_3$, since

$$WH^+ = \begin{bmatrix} w & 0^T \\ 0 & O_2 \end{bmatrix}\begin{bmatrix} 0 & (EM)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix} = \begin{bmatrix} 0 & w(EM)^+E \\ 0 & O_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & w(EM)^+(I_2-MM^+) \\ 0 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & wE^+M^+ \\ 0 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0^T \\ 0 & O_2 \end{bmatrix},$$

and

$$H^+W = \begin{bmatrix} 0 & (EM)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix}\begin{bmatrix} w & 0^T \\ 0 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0^T \\ F(h^TF)^+w & O_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0^T \\ F(h^TF)^+ysx & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0^T \\ 0 & O_2 \end{bmatrix}.$$

Now, if $\tilde{Z}$ is regular, then $W$ is regular, by Lemma 2.2. This implies that $w$ is regular.

Conversely, assume that $w$ is regular. Let

$$X = \begin{bmatrix} xw^-y & (Ev)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix}.$$

We claim that $X$ is a {1}- inverse of $\tilde{Z}$ Consider,

$$\tilde{Z}X\tilde{Z} = \begin{bmatrix} w & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} xw^- w + 1 - x & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix} = \tilde{Z}.$$

Now,

$$X\tilde{Z} = \begin{bmatrix} xw^- y & (Ev)^+ E \\ F(h^T F)^+ & M^+ \end{bmatrix} \begin{bmatrix} ysx & h^T F \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} xw^- yysx + (Ev)^+ EEv & xw^- yh^T F + (Ev)^+ EM \\ F(h^T F)^+ ysx + M^+ Ev & F(h^T F)^+ h^T F + M^+ M \end{bmatrix}$$

$$= \begin{bmatrix} xw^- ysx + (Ev)^+ Ev & 0^T \\ 0 & F(h^T F)^+ h^T F + M^+ M \end{bmatrix}$$

$$= \begin{bmatrix} xw^- w + (Ev)^+ Ev & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix}$$

$$= \begin{bmatrix} xw^- w + 1 - x & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix}.$$

Indeed,

$$\tilde{Z}X\tilde{Z} = \begin{bmatrix} ysx & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} xw^- w + 1 - x & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix} = \tilde{Z}$$

$$= \begin{bmatrix} ysx(xw^- w + 1 - x) & h^T F(M^+ M + F(h^T F)^+ h^T F) \\ Ev(xw^- w + 1 - x) & M(M^+ M + F(h^T F)^+ h^T F) \end{bmatrix}$$

$$= \begin{bmatrix} ysxxw^- w + ysx - ysxx & h^T FM^+ M + h^T FF(h^T F)^+ h^T F \\ Evxw^- w + Ev - Evx & MM^+ M + MF(h^T F)^+ h^T F \end{bmatrix}$$

$$= \begin{bmatrix} ysxw^- w + ysx - ysx & h^T F(h^T F)^+ h^T F \\ Ev(1 - (Ev)^+ Ev)w^- w - Ev(1 - (Ev)^+ Ev) + Ev & MM^+ M \end{bmatrix}$$

$$= \begin{bmatrix} ysxw^- w & h^T F \\ Evw^- w - Ev(Ev)^+ Evw^- w - Ev + Ev(Ev)^+ Ev + Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} ww^- w & h^T F \\ Evw^- w - Evw^- w - Ev + Ev + Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} w & h^T F \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} ysx & h^T F \\ Ev & M \end{bmatrix}.$$

In view of (3.2) we conclude that a {1}- inverse of $Z$ is given by (3.3) and, thus, $Z$ is regular. It remains to prove (3.4) but the proof of this is straightforward. In fact, for $Z$ in (2.3) we have

$$Z^- = \begin{bmatrix} 1 & 0^T \\ -F(h^T F)^+ s & I_2 \end{bmatrix} \begin{bmatrix} xw^- y & (Ev)^+ E \\ -F(h^T F)^+ & M^+ \end{bmatrix} \begin{bmatrix} 1 & -ys(Ev)^+ E \\ 0 & I_2 \end{bmatrix} \qquad (3.3)$$

is a {1}- inverse of $Z$. By direct computation we have

$$I_3 - ZZ^- = \begin{bmatrix} (1 - ww^-)y & -(1 - ww^-)ys(Ev)^+ E \\ 0 & (1 - Ev(Ev)^+)E \end{bmatrix}. \qquad (3.4)$$

From, our assumption, we have $M$ is the group inverse. Then we can set $M^+ = M^\#$. In this case, in the notation of (2.2) we have $E = I_2 - MM^\# = I_2 - M^\#M = F$. It follows that $ME = EM^\# = O_2$, since

$$ME = M(I_2 - M^\#M) = M - MM^\#M = M - M = O_2,$$
$$EM^\# = (I_2 - M^\#M)M^\# = M^\# - M^\#MM^\# = M^\# - M^\# = O_2.$$

We claim that $M + E$ is a unit of $R^{2\times2}$ and

$$(M + E)^{-1} = M^\# + E.$$

For,

$$\begin{aligned}(M + E)(M^\# + E) &= M(M^\# + E) + E(M^\# + E)\\ &= MM^\# + ME + EM^\# + EE\\ &= MM^\# + O_2 + O_2 + E\\ &= MM^\# + (I_2 - MM^\#)\\ &= O_2,\end{aligned}$$

and

$$\begin{aligned}(M^\# + E)(M + E) &= M^\#(M + E) + E(M + E)\\ &= M^\#M + M^\#E + EM + EE\\ &= M^\#M + O_2 + O_2 + E\\ &= M^\#M + (I_2 - M^\#M)\\ &= O_2.\end{aligned}$$

$\square$

*Theorem* 3.2. *Let* $M$ *be group invertible. With the notation* (2.2) *and under the assumptions of* (3.1), *with* $M^+$ *replaced by* $M^\#$ *if* $w$ *is regular in* $R$ *then the group inverse of the matrix*

$$N = \begin{bmatrix} e & h_1 & h_2 \\ v_1 & a & c \\ v_2 & b & d \end{bmatrix} = \begin{bmatrix} e & h^T \\ v & M \end{bmatrix}$$

*exists if and only if*

$$r = sx - h^T Ev + (1 - ww^-)y[s + (1 + h^T(M^\#)^2 v)x] \quad (3.5)$$

*is a unit of* $R$. *In this case,*

$$N^\# = \begin{bmatrix} \omega_1 & \Omega_3 \\ \Omega_2 & \Omega_4 \end{bmatrix},$$

*where*

$$\begin{aligned}\omega_1 &= \gamma - x(\lambda\gamma - r^{-1}),\\ \Omega_2 &= -M^\#(v - M^\#vx)\gamma + (M^\#vx + Ev)(\lambda\gamma - r^{-1}),\\ \Omega_3 &= -\Theta + x(\gamma h^T(M^\#)^2 + \lambda\Theta - r^{-1}h^T M^\#),\\ \Omega_4 &= M^\# + M^\#(v - M^\#vx)\Theta - (M^\#vx + Ev)(\gamma h^T(M^\#)^2 + \lambda\Theta - r^1 h^T M^\#),\end{aligned} \quad (3.6)$$

*with*

$$\begin{aligned}\gamma &= r^{-1}(1 - ww^-)y,\\ \Theta &= \gamma h^T M^\# + r^{-1}h^T E,\\ \lambda &= \gamma(1 + h^T(M^\#)^2(v - M^\#vx)) + r^{-1}(s + (1 + h^T(M^\#)^2 v)x).\end{aligned} \quad (3.7)$$

**Proof:** Write $N = XZY$ as in (2.3), by Proposition 2.4, the group inverse of $N$ exists if and only if $T = ZYX + I_3 - ZZ^-$ is a unit, independent of the choice of $Z^-$. For the {1}-inverse provided in Lemma 3.1, we have

$$T = ZYX + I_3 - ZZ^- = \begin{bmatrix} s + (1 - ww^-)y & sh^T M^\# + h^T E - (1 - ww^-)ys(Ev)^+ E \\ v & M + vh^T M^\# + (1 - Ev(Ev)^+)E \end{bmatrix}.$$

Now, let us introduce the matrix

$$G = \begin{bmatrix} 1 & (Ev)^+ E - h^T M^\# \\ 0 & I_2 \end{bmatrix}.$$

We have

$$TG = \begin{bmatrix} s + (1 - ww^-)y & u^T \\ v & K \end{bmatrix}, \tag{3.8}$$

where

$$K = M + E + MM^\# v(Ev)^+ E = (M + E)(1 + M^\# v(Ev)^+ E),$$
$$u = (s + (1 - ww^-)y)((Ev)^+ E - h^T M^\#) + (sh^T M^\# + h^T E - (1 - ww^-)ys(Ev)^+ E). \tag{3.9}$$

Since the element $M^\# v(Ev)^+ E$ is 2-nilpotent, because

$$(M^\# v(Ev)^+ E)(M^\# v(Ev)^+ E) = M^\# v(Ev)^+ EM^\# v(Ev)^+ E = M^\# v(Ev)^+ 0v(Ev)^+ E = O.$$

It follows that $1 + M^\# v(Ev)^+ E$ is a unit of $R$.

$$(1 + M^\# v(Ev)^+ E)(1 - M^\# v(Ev)^+ E) = ((1 - M^\# v(Ev)^+ E) + M^\# v(Ev)^+ E(1 - M^\# v(Ev)^+ E))$$
$$= 1 - M^\# v(Ev)^+ E + M^\# v(Ev)^+ E - M^\# v(Ev)^+ EM^\# v(Ev)^+ E$$
$$= 1 - M^\# v(Ev)^+ 0v(Ev)^+ E$$
$$= 1.$$

Moreover $M + E$ is a unit because $M$ has group inverse. Thus, $K$ is a unit and

$$S = K^{-1} = (1 - M^\# v(Ev)^+ E)(M^\# + E). \tag{3.10}$$

On account that the element $(2,2)$ of the matrix $TG$ is a unit, it follows that $TG$ is a unit of $R^{2\times 2}$ if and only if the Schur complement is a unit of $R$. Therefore, the matrix $T$ is a unit if and only if

$$r = s + (1 - ww^-)y - uSv \tag{3.11}$$

is a unit in $R$. From (3.10), we get $Sv = Ev + M^\# vx.$

Now,

$$Sv = [(1 - M^\# v(Ev)^+ E)(M^\# + E)]v$$
$$= [(M^\# + E) - M^\# v(Ev)^+ E(M^\# + E)]v$$
$$= [M^\# + E - M^\# v(Ev)^+ EM^\# - M^\# v(Ev)^+ EE]v$$
$$= M^\# v + Ev - M^\# v(Ev)^+ 0v - M^\# v(Ev)^+ EEv$$
$$= Ev + M^\# v - M^\# v(Ev)^+ EEv$$
$$= Ev + M^\# v(1 - (Ev)^+ Ev)$$
$$= Ev + M^\# vx.$$

Further, using that last relation of (3.9) we obtain

$$uSv = s(1 - x) + h^T Ev - (1 - ww^-)y[(s - 1) + (1 + h^T(M^\#)^2 v)x].$$

By substituting this expression in (3.11), we conclude that $r$ has the form given in (3.5). By Proposition 2.4, $N^\# = XT^{-2}ZY.$ From (3.8), it follows that

$$(TG)^{-1} = \begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}.$$

Next, we compute

$$XT^{-1} = XG(TG)^{-1} = \begin{bmatrix} 1 & h^T M^{\#} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & (Ev)^{+}E - h^T M^{\#} \\ 0 & 1 \end{bmatrix}(TG)^{-1}$$

$$= \begin{bmatrix} xr^{-1} & (Ev)^{+}E - xr^{-1}uS \\ -Sr^{-1} & S + Svr^{-1}uS \end{bmatrix}. \tag{3.12}$$

Now,

$$XT^{-1} = XG(TG)^{-1} = \begin{bmatrix} 1 & h^T M^{\#} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & (Ev)^{+}E - h^T M^{\#} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (Ev)^{+}E - h^T M^{\#} + h^T M^{\#} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (Ev)^{+}E \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

$$= \begin{bmatrix} r^{-1} - (Ev)^{+}ESvr^{-1} & -r^{-1}uS + (Ev)^{+}E(S + Svr^{-1}uS) \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

$$= \begin{bmatrix} r^{-1} - (Ev)^{+}ESvr^{-1} & -r^{-1}uS + (Ev)^{+}ES + (Ev)^{+}ESvr^{-1}uS \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

$$= \begin{bmatrix} r^{-1} - (Ev)^{+}Evr^{-1} & (Ev)^{+}E - r^{-1}uS + (Ev)^{+}Evr^{-1}uS \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}; ES = E$$

$$= \begin{bmatrix} (1 - (Ev)^{+}Ev)r^{-1} & (Ev)^{+}E - (1 - (Ev)^{+}Ev)r^{-1}uS \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

$$= \begin{bmatrix} xr^{-1} & (Ev)^{+}E - xr^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

the last equality is due to the fact that $ES = E$. In the sequel, we denote

$$\gamma = r^{-1}(1 - ww^{-})y \qquad \text{and} \qquad \Theta = \gamma h^T M^{\#} + r^{-1}h^T E.$$

From (3.10), (3.9), and (3.11) it follows that

$$SM = M^{\#}M,$$
$$r^{-1}uM^{\#}M = -\gamma h^T M^{\#}, \tag{3.13}$$
$$r^{-1}uSv = r^{-1}(s + (1 - ww^{-})y) - 1 = r^{-1}s + \gamma - 1$$

respectively. In deriving the last equality, we have multiplied on the left expression (3.11) by $r^{-1}$. Then

$$T^{-1}ZY = G(TG)^{-1}ZY$$

$$= G(TG)^{-1} \begin{bmatrix} s & h^T E \\ Ev & M \end{bmatrix} \begin{bmatrix} 1 & 0 \\ M^{\#}v & I_2 \end{bmatrix}$$

$$= G \begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix} \begin{bmatrix} s & h^T E \\ v & M \end{bmatrix} \qquad (3.14)$$

$$= \begin{bmatrix} 1 & (Ev)^+ E - h^T M^{\#} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\gamma & \Theta \\ Sv\gamma & M^{\#}M - Sv\Theta \end{bmatrix}$$

$$= \begin{bmatrix} 1-(1+h^T(M^{\#})^2v)x\gamma & -h^T M^{\#} + (1+h^T(M^{\#})^2v)x\Theta \\ Sv\gamma & M^{\#}M - Sv\Theta \end{bmatrix}.$$

Using (3.12) and (3.14), we obtain

$$XT^{-1}ZY = \begin{bmatrix} xr^{-1} & (Ev)^+E - xr^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix} \begin{bmatrix} 1-(1+h^T(M^{\#})^2v)x\gamma & -h^T M^{\#} + (1+h^T(M^{\#})^2v)x\Theta \\ Sv\gamma & M^{\#}M - Sv\Theta \end{bmatrix}$$

$$= \begin{bmatrix} \omega_1 & \Omega_3 \\ \Omega_2 & \Omega_4 \end{bmatrix},$$

where

$$\omega_1 = (1-x)\gamma - xr^{-1}(uS^2v\gamma - 1 + (1+h^T(M^{\#})^2v)x\gamma),$$

$$\Omega_2 = S^2v\gamma + uvr^{-1}(uS^2v\gamma - 1 + (1+h^T(M^{\#})^2v)x\gamma),$$

$$\Omega_3 = (x-1)\Theta + xr^{-1}(uS^2v\Theta - uM^{\#} - h^T M^{\#} + (1+h^T(M^{\#})^2v)x\Theta), \qquad (3.15)$$

$$\Omega_4 = M^{\#} - S^2v\Theta - Svr^{-1}(uS^2v\Theta - uM^{\#} - h^T M^{\#} + (1+h^T(M^{\#})^2v)x\Theta).$$

From (3.10), it follows that $S^2v = Sv - M^{\#}(v - M^{\#}vx).$

$$S^2v = S(Sv) = S(Ev + M^{\#}vx)$$

$$= S(E)v + SM^{\#}vx$$

$$= S(1 - M^{\#}M)v + SM^{\#}vx$$

$$= Sv - SM^{\#}Mv + SM^{\#}vx$$

$$= Sv - M^{\#}v + ((1 - M^{\#}v(Ev)^+ E)(M^{\#} + E))M^{\#}vx$$

$$= Sv - M^{\#}v + ((M^{\#}M^{\#}vx + EM^{\#}vx) - M^{\#}v(Ev)^+ EEM^{\#}vx)$$

$$= Sv - M^{\#}v + M^{\#}M^{\#}vx + EM^{\#}vx - M^{\#}v(Ev)^+ EEM^{\#}vx$$

$$= Sv - M^{\#}v + M^{\#}M^{\#}vx$$

$$= Sv - M^{\#}(v - M^{\#}vx).$$

Using this latter expression and (3.13) we get

$$r^{-1}uS^2v = r^{-1}uSv - r^{-1}uM^{\#}(v - M^{\#}vx) = r^{-1}s + \gamma - 1 + \gamma h^T(M^{\#})^2(v - M^{\#}vx).$$

By substituting this into (3.15), using $r^{-1}uM^{\#} = -\gamma h^T(M^{\#})^2,$ and regrouping terms we get (3.6).

$\square$

## Conclusions

In this work we have studied conditions for the existence of the group inverse of the $3 \times 3$ matrix over a ring with unity $1$ under some development certain conditions.

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## References

Ben-Israel, A., & Greville, N. E. (2003). *Generalized Inverses, Theory and Applications, Second Edition*. New York: Springer.

Bhaskara Rao, K. P. S. (2002). *The Theory of Generalized Inverses over Commutative Rings*. London and New York: Taylor & Francis.

Cao, C., Ge, Y., Wang, Y., & Zhang, H. (2013). Group inverse for two classes of $2 \times 2$ block matrices over rings. *Comp. Appl. Math.*, *2014*(33), 469-479.

Castro-Gonzalez, N., Robles, J., & Velez-Cerrada, J. Y. (2013). The group inverse of $2 \times 2$ matrices over a ring. *Linear Algebra and its Applications*, 438, 3600-3609.

Catral, M., Olesky, D. D. & Driessche, P. (2008). Group inverse of matrices with path graphs. *Electronic Journal of Linear Algebra (ELA)*, 11, 219-233.

Heinig, G. (1997). The group inverse of the transformation $\mathfrak{S}(X) = AX - XB,$. *Linear Algebra and its Applications*, 257, 321-342.

Puystjens, R., & Hartwig, R. E. (1997). The group inverse of a companion matrix. *Linear and Multilinear Algebra*, 43, 137-150.

Puystjens, R., & Hartwig, R. E. (2010). The (2, 2, 0) group inverse problem. *Appl. Math. Comput*, 217, 516-520.

Rajesh Kannan, M., & Bapat, R. B. (2014). Generalized principal pivot transform. *Linear Algebra and its Applications*, 454, 49-56.

Wei, Y., & Diao, H., (2005). On group inverse of singular Toeplitz matrices. *Linear Algebra and its Applications*, 399, 109-123.