Application of Laplace Differential Transform Method in Solving Two-Dimensional Partial Differential Equations with Variable Coefficient

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Abstract

In this paper, Laplace Differential Transform Method (LDTM) is employed in solving two-dimensional partial differential equations with variable coefficients. Laplace Differential Transform Method (LDTM) combines Laplace transform and Differential Transform Method (DTM) and can be used to effectively solve 2-D partial differential equations. In order to demonstrate the effectiveness of this method, 2-D heat-like equations and wave-like equation were considered. Results revealed that the LDTM is effective and efficient in handling 2-D homogeneous and nonhomogeneous partial differential equations with little computational effort.

Keywords: Nonhomogeneous PDE, 2-D PDE, Laplace Differential Transform Method, Laplace Transform, Differential Transform Method

Introduction

Evolution equations have attracted a lot of attention over the past five decades due to its broad applications in areas such as Biomathematics, engineering, fluid dynamics, chemistry etc. Although most of these equations do not have an analytical solution, researchers have either been using numerical or approximate methods to solve these problems. However, these methods are computationally intensive as it involves a lot of numerical computations, approximations and manipulations. Consequently, researchers are recently working to develop new methods to solve these problems.

Among these methods, we have Homotopy Perturbation Method (Alquran & Mohammad, 2011; He, 2006; Moghimi, Ganji, Bararnia, Hosseini, & Jalaal, 2011), Homotopy Analysis Method (Islam, Khan, Faraz, & Austin, 2010; Jafari, Chun, Seifi, & Saeidy 2009) and Differential transform method (Al-Ahmad, Mamat, & AlAhmad, 2020; Ganji, Jouya, Mirhosseini-Amiri, & Ganji, 2016; Ghafoori et al., 2011; Zou, Zong, Wang, & Wang, 2010) to mention a few. The aforementioned methods have effectively been used to solve one-dimensional evolution equations. But, when it comes to two-dimensional evolution equations, it's very tedious, so, most studies in the literature focus on one-dimensional evolution equations. Although, Differential transform method has been used to solve two dimensional PDEs. The concept of Differential transform was first introduced by Zhou (1986) and applied to solve initial value problems in electric circuit analysis. Many researchers have contributed to literature using this method (Al-Ahmmad et al., 2020; Ganji et al., 2016; Ghafoori et al., 2011; Zou et al., 2010). However, using Differential transform to solve two-dimensional evolution partial differential equations becomes more complicated. Hence, the need to combine another method to effectively solve any two-dimensional evolution partial differential equations.

In this paper, Laplace Differential transform method (LDTM), which combines Laplace transform and Differential Transform Method (DTM), is used to solve the 2-dimensional heat-like equation and wave-like equation, with the aim of showing the reliability of this method in proffering an exact solution of two dimensional homogeneous and non-homogenous PDEs.

The present paper has been organized as follows: in section 2, basic definitions of DTM and the basic idea of LDTM. In section 3, we present three examples in form of 2-D heat-like equation and wave-like equation. In section 4, the conclusion follows.

Methods and Materials

Solution Formulations and Definitions

In this section, basic definitions of semi-analytical method DTM for better understanding of LDTM are introduced as follows:

Overview of 2D Differential Transform Method (DTM)

The differential transform method constructs a semi-analytical numerical technique that applies Taylor series to the solution of differential equations in the form of polynomials.

Considering $\overline{w}(x,y)$ as an analytical and continuously differentiable function, the differential transformation of the $k+h$th derivative of the function $\overline{w}(x,y)$ is defined as follows:

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}\overline{w}(x,y)}{\partial x^k \partial y^h}\right]_{\substack{x=x_0\\ y=y_0}} \tag{1}$$

Where $\overline{w}(x,y)$ is the original function and $W(k,h)$, the spectrum function is the transformed function. The differential inverse transformation of $W(k,h)$ can be defined in the following form:

$$\overline{w}(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)(x-x_0)^k(y-y_0)^h \tag{2}$$

Substituting equation (1) into equation (2), gives

$$\overline{w}(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}\overline{w}(x,y)}{\partial x^k \partial y^h}\right]_{\substack{x=x_0\\ y=y_0}}(x-x_0)^k(y-y_0)^h \tag{3}$$

Equating (x_0,t_0) as $(0,0)$, equation (2) can be written in the form:

$$\overline{w}(x,y)=\sum_{k=0}^{n}\sum_{h=0}^{m}W(k,h)x^k y^h \tag{4}$$

The function $\overline{w}(x,y)$ can be represented by a finite series:

$$\overline{w}(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k y^h = \sum_{k=0}^{a}\sum_{h=0}^{b}W(k,h)x^k y^h + R_{kh}(x,y) \tag{5}$$

Where

$$R_{kh}(x,y)=\sum_{k=a+1}^{\infty}\sum_{h=b+1}^{\infty}W(k,h)x^k y^h \tag{6}$$

R_{kh} is the remainder term, for which the values of a and b are obtained by convergence of the series coefficients.

Property 1(Two-Dimensional Differential Transform Properties): The basic operations of the two dimensional transform which are useful in the transformation in this paper are summarized as follows: (Ganji et al., 2016; Tari & Shahmorad, 2011)

	Original Function	2D Transformed Function
(a)	$f(x,y)=\dfrac{\partial \overline{w}(x,y)}{\partial x}$	$\overline{F}=(k+1)W(k+1,h)$
(b)	$f(x,y)=\dfrac{\partial \overline{w}(x,y)}{\partial y}$	$\overline{F}=(h+1)W(k,h+1)$
(c)	$f(x,y)=\dfrac{\partial^{i+j} \overline{w}(x,y)}{\partial x^i \partial y^j}$	$\overline{F}(k,h)=\dfrac{(k+i)!(h+j)!}{k!h!}W(k+i,h+j)$
(d)	$f(x,y)=\lambda\overline{w}(x,y)$	$\overline{F}(k,h)=\lambda W(k,h)$
(e)	$f(x,y)=x^i y^j$	$\overline{F}(k,h)=\delta(k-i,h-j)=\delta(k-i)\delta(h-j) = \begin{cases}1, & k=i, h=j\\ 0, & \text{otherwise}\end{cases}$
(f)	$f(x,y)=\overline{u}(x,y)\overline{v}(x,y)$	$\overline{F}(k,h)=\sum_{j=0}^{h}\sum_{i=0}^{k}U(i,h-j)V(k-i,h)$
(g)	$f(x,y)=\overline{u}(x,y)\pm\overline{v}(x,y)$	$\overline{F}(k,h)=U(k,h)\pm V(k,h)$

Basic Idea of Two-dimensional LDTM

To illustrate the basic idea of this method, we consider the general form of two-dimensional second order nonhomogeneous partial differential equations with variable coefficients of the form:

$$\frac{\partial^2 w(x,y,t)}{\partial t^2}+a_n(x,y)Rw(x,y,t)=f(x,y,t),\ \ t>0, x>0, y<1, n\in \mathrm{N} \tag{7}$$

Where $a_n(x,y)$ is the variable coefficients, $n \in \mathbb{N}$, R is the linear operator and $f(x,y,t)$ is the source function.

With the initial conditions

$$w(x,y,0)=g_1(x,y),\quad w_t(x,y,0)=g_2(x,y) \tag{8}$$

The method involves applying a Laplace transform to equation (7) -(8) and the use of the linearity property of Laplace transform

$$\mathrm{L}\left\{\frac{\partial^2 w(x,y,t)}{\partial t^2}\right\} + \mathrm{L}\left\{a_n(x,y)Rw(x,y,t)\right\} = \mathrm{L}\left\{f(x,y,t)\right\}$$

$$s^2\overline{w}(x,y,s) - s\overline{w}(x,y,0) - \overline{w}_t(x,y,0) + a_n(x,y)R\overline{w}(x,y,s) = \overline{f}(x,y,s) \quad (9)$$

Where $\overline{w}(x,y,s)$ is the Laplace transform of $w(x,y,t)$.

Applying the initial conditions in equation (8) to equation (9) gives

$$s^2\overline{w}(x,y,s) - sg_1(x,y) - g_2(x,y) + a_n(x,y)R\overline{w}(x,y,s) = \overline{f}(x,y,s) \quad (10)$$

Accordingly, using differential transform method, the solution of equations (10) can be written as:

$$\overline{w}(x,y,s) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h \quad (11)$$

Where $W(k,h) = \frac{1}{k!h!}\left(\frac{\partial^{k+h} w(x,y,s)}{\partial x^k \partial y^h}\right), \qquad k,h = 0,1,2,\ldots$

$W(k,h)$ represents the differential transform of $\overline{w}(x,y,s)$ and $W(k,h)$ is a function of the parameter s. After determining $\overline{w}(x,y,s)$, inverse Laplace transform is applied to equation (11) to get $w(x,y,t)$ that is

$$w(x,y,t) = \mathrm{L}_s^{-1}\left\{\overline{w}(x,y,s)\right\} = \mathrm{L}_s^{-1}\left\{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h\right\} \quad (12)$$

Application of this Method

Here, Illustrative examples are considered to demonstrate the applicability of using Laplace Differential Transform Method (LDTM) in solving 2 dimensional heat-like equation and wave-like equation.

Example 1

Consider the following 2-dimensional heat-like equation with variable coefficient given by (Neog, 2015):

$$\frac{\partial w(x,y,t)}{\partial t} = \frac{1}{2}\left[x^2\frac{\partial^2 w(x,y,t)}{\partial x^2} + y^2\frac{\partial^2 w(x,y,t)}{\partial y^2}\right], \qquad x>0, y<1, t>0 \quad (13)$$

With the initial condition

$$w(x,y,0) = x^2 + y^2 \quad (14)$$

Applying Laplace transform to equation (13) in view of boundary condition (14) gives

$$s\overline{w} - x^2 - y^2 = \frac{1}{2}x^2\overline{w}_{xx} + \frac{1}{2}y^2\overline{w}_{yy} \tag{15}$$

Where $\overline{w} = \overline{w}(x, y, s)$

Applying the differential transform in Property 1: (a), (b), (e) to (15) gives

$$sW(k,h) - \delta(k-2)\delta(h) - \delta(h-2)\delta(k) = \frac{1}{2}\sum_{j=0}^{h}\sum_{i=0}^{k}(i+1)(i+2)W(i+2,j)\delta(h-j)\delta(k-i-2)$$
$$+\frac{1}{2}\sum_{j=0}^{h}\sum_{i=0}^{k}(j+1)(j+2)W(i,j+2)\delta(h-j-2)\delta(k-i)$$

(16)

Where $W(k,h) = \frac{1}{k!h!}\left(\frac{\partial^{k+h}\overline{w}(x,y,s)}{\partial x^k \partial y^h}\right), \qquad k,h = 0,1,2,\ldots$

Substitute for $k = 0$ in (16) gives

$$sW(0,h) - \delta(h-2) = \sum_{j=0}^{h}(j+1)(j+2)\delta(h-j-2)W(0,j+2) \tag{17}$$

Substitute for $h = 0$ in (16) gives

$$sW(k,0) - \delta(k-2) = \sum_{i=0}^{k}(i+1)(i+2)\delta(k-i-2)W(i+2,0) \tag{18}$$

From (17), we obtain the following results:

$$\begin{aligned} &W(0,0) = 0 \\ &W(0,1) = 0 \\ &sW(0,2) - 1 = W(0,2) \\ &W(0,h) = 0 \text{ for } h \geq 3 \end{aligned} \tag{19}$$

Likewise from (18), we obtain the following results:

$$\begin{aligned} &W(0,0) = 0 \\ &W(1,0) = 0 \\ &sW(2,0) - 1 = W(2,0) \\ &W(k,0) = 0, \text{ for k} \geq 3 \end{aligned} \tag{20}$$

Further algebraic evaluation using equations (19) and (20), we arrive at

$$W(0,2) = W(2,0) = \frac{1}{s-1} \tag{21}$$

By the equations (16), (19), (20) and the use of the property of $\delta(h), \delta(k)$ in Property 1, we have that

$$W(h,k) = 0: \qquad h,k = 1,2,3,\ldots$$

Hence,

$$\overline{w}(x,y,s) = W(2,0)x^2 + W(0,2)y^2$$
$$= (x^2 + y^2)\frac{1}{s-1} \tag{22}$$

Applying the Laplace inverse i.e,

$$w(x,y,t) = \mathrm{L}_s^{-1}\left[\overline{w}(x,y,s)\right]$$
$$= \mathrm{L}_s^{-1}\left[(x^2+y^2)\frac{1}{s-1}\right]$$

gives

$$w(x,y,t) = (x^2+y^2)e^t \tag{23}$$

Equation (23) is the exact solution of (13) - (14) as exactly obtained in Neog (2015).

Example 2

Consider the following two dimensional nonhomogeneous heat-like equation with variable coefficient given by (Neog, 2015):

$$\frac{\partial w(x,y,t)}{\partial t} = x^2y^2 + \frac{1}{4}\left[x^2\frac{\partial^2 w(x,y,t)}{\partial x^2} + y^2\frac{\partial^2 w(x,y,t)}{\partial y^2}\right], \qquad x>0, y<1, t>0 \tag{24}$$

With the initial condition

$$w(x,y,0) = 0 \tag{25}$$

The Laplace transform of equation (24) in view of the initial condition (25) gives

$$s\overline{w} = x^2y^2 + \frac{1}{4}\left[x^2\frac{\partial^2\overline{w}}{\partial x^2} + y^2\frac{\partial^2\overline{w}}{\partial y^2}\right] \tag{26}$$

Where $\overline{w} = \overline{w}(x,y,s)$

Applying the differential transform in Property 1: (a), (b), (e) to (26) gives

$$sW(k,h) = \delta(k-2,h-2) + \frac{1}{4}\sum_{j=0}^{h}\sum_{i=0}^{k}(i+1)(i+2)W(i+2,j)\delta(k-i-2)\delta(h-j)$$
$$+\frac{1}{4}\sum_{j=0}^{h}\sum_{i=0}^{k}(j+1)(j+2)W(i,j+2)\delta(k-i)\delta(h-j-2) \tag{27}$$

Where $W(k,h) = \frac{1}{k!h!}\left(\frac{\partial^{k+h}\overline{w}(x,y,s)}{\partial x^k \partial y^h}\right), \qquad k,h = 0,1,2,\ldots$

Substitute for $k=2$ in equation (27) gives

$$sW(2,h) = \delta(0,h-2) + \frac{1}{2}\sum_{j=0}^{h}W(2,j)\delta(h-j) + \frac{1}{4}\sum_{j=0}^{h}(j+1)(j+2)W(2,j+2)\delta(h-j-2) \tag{28}$$

Substitute for $h=2$ in equations (27) gives

$$sW(k,2)=\delta(k-2,0)+\frac{1}{2}\sum_{i=0}^{k}W(i,2)\delta(k-i)+\frac{1}{4}\sum_{i=0}^{k}(i+1)(i+2)W(i+2,2)\delta(k-i-2) \quad (29)$$

From equations equations (28) and (29), we have the following results:

$$\begin{aligned} W(0,0)&=0,\\ W(0,1)&=0,\\ W(1,0)&=0,\\ W(1,1)&=0,\\ W(2,1)&=0,\\ W(1,2)&=0,\\ W(2,2)&=\frac{1}{s-1} \end{aligned} \quad (30)$$

By the use of the results in equation (30), all other coefficients of the DT series are all zero.

Therefore,

$$\begin{aligned} \overline{w}(x,y,s)&=W(2,2)x^2y^2\\ &=\frac{1}{s-1}x^2y^2\\ w(x,y,t)&=L_s^{-1}\left[\overline{w}(x,y,s)\right] \end{aligned}$$

$$w(x,y,t)=x^2y^2e^t \quad (31)$$

Equation (31) is the exact solution of (24)-(25), which is exactly the same as that obtained in Neog (2015).

Example 3

Consider the following two-dimensional nonhomogeneous wave-like equation with variable coefficient:

$$\frac{\partial^2 w(x,y,t)}{\partial t^2}=x^5+\frac{1}{20}\left[x^2\frac{\partial^2 w(x,y,t)}{\partial x^2}+y^2\frac{\partial^2 w(x,y,t)}{\partial y^2}\right], \qquad x>0, y<1, t>0 \quad (32)$$

With the initial conditions

$$w(x,y,0)=y^5, \qquad \frac{\partial w(x,y,0)}{\partial t}=0 \quad (33)$$

In view of the initial conditions (33), the Laplace transform of equation (32) is

$$s^2\overline{w}-sy^5=\frac{x^5}{s}+\frac{1}{20}x^2\overline{w}_{xx}+\frac{1}{20}y^2\overline{w}_{yy} \quad (34)$$

Where $\overline{w}=\overline{w}(x,y,s)$

Applying the differential transform in property 1: (a), (b), (c) to equation (34) gives

$$\begin{aligned} s^2W(k,h)-s\delta(h-5)\delta(k)=\frac{1}{s}\delta(k-5)\delta(h)&+\frac{1}{20}\sum_{j=0}^{h}\sum_{i=0}^{k}(i+1)(i+2)W(i+2,j)\delta(k-i-2)\delta(h-j)\\ &+\frac{1}{20}\sum_{j=0}^{h}\sum_{i=0}^{k}(j+1)(j+2)W(i,j+2)\delta(k-i)\delta(h-j-2) \end{aligned} \quad (35)$$

Where $W(k,h)=\dfrac{1}{k!h!}\left(\dfrac{\partial^{k+h}\overline{w}(x,y,s)}{\partial x^k \partial y^h}\right), \qquad k,h=0,1,2,\ldots$

Substitute for $k=0$ in equation (35) gives

$$s^2W(0,h)-s\delta(h-5)=\frac{1}{20}\sum_{j=0}^{h}\delta(h-j-2)(j+1)(j+2)W(0,j+2) \qquad (36)$$

Substitute for $h=0$ in equation (35) gives

$$s^2W(k,0)=\frac{1}{s}\delta(k-5)+\frac{1}{20}\sum_{i=0}^{k}\delta(k-i-2)(i+1)(i+2)W(i+2,0) \qquad (37)$$

From equations (36) and (37), we obtain the following results

$$\begin{aligned}
&W(0,0)=0,\\
&W(1,0)=W(0,1)=0,\\
&W(2,0)=W(0,2)=0,\\
&W(3,0)=W(0,3)=0,\\
&W(4,0)=W(0,4)=0,\\
&W(5,0)=\frac{1}{s(s^2-1)},\\
&W(0,5)=\frac{s}{s^2-1},\\
&W(k,0)=W(0,h)=0 \text{ for } h,k\geq 6
\end{aligned} \qquad (38)$$

By the use of the results in equation (38), all other coefficients of the DT series are all zero.

Hence,

$$\overline{w}(x,y,s)=W(5,0)x^5+W(0,5)y^5$$

$$\begin{aligned}
&\overline{w}(x,y,s)=\frac{1}{s(s^2-1)}x^5+\frac{s}{s^2-1}y^5\\
&w(x,y,t)=\mathrm{L}_s^{-1}\left[\overline{w}(x,y,s)\right]
\end{aligned} \qquad (39)$$

$$w(x,y,t)=(x^5+y^5)\cosh t-x^5 \qquad (40)$$

Equation (40) is the exact solution of (32)-(33).

Conclusion

The applicability of the combined form of Laplace transform method and Differential Transform Method (LDTM) is demonstrated in this paper, to solve two-dimensional homogeneous and nonhomogeneous of heat like equations and wave-like equation with variable coefficients. The combined methods efficiently give the exact solution of two dimensional partial differential equations with little computational work. Laplace Differential Transform Method (LDTM) can be applied to solve other two-dimensional homogeneous & nonhomogeneous partial differential equations and does not require linearization, discretization or perturbation.

This study considered only two dimensional partial differential equations, for a higher dimensions, it may become complicated and tedious to solve. Hence, for future study, the condition for which this method can be applied to solve for higher dimensions of partial differential equations should be included.

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