On bi-bases of ordered Γ-semigroups

Wichayaporn Jantanan, Monthean Latthi and Jiriya Puifai*

Department of Mathematics Faculty of Science Buriram Rajabhat University Buriram 31000

* Corresponding author. E-mail address: wichayaporn.jan@bru.ac.th; 600112210028@bru.ac.th

Received: 16 August 2020; Revised: 9 August 2021; Accepted: 16 August 2021; Available online: 28 October 2021

Abstract

In this paper, based on the results of ordered bi-ideals generated by a non-empty subset of an ordered Γ-semigroups M, we introduce the concept of bi-base of M. Using the quasi-order on M defined by the principal ordered bi-ideals of M. we characterize when a non-empty subset of M is a bi-base of M. The results obtained extending the results on Γ-semigroup.

Keywords: ordered Γ-Semigroup, bi-Γ-ideal, bi-base, quasi-order

Introduction

The notion of two-sided bases of a semigroup was introduced by Fabrici (1975). The results (Fabrici, 1975). Have extended to ordered semigroups by Changpas and Summaprab (Changpas&Summaprab, 2014). In 2017 Kummoon and Changpas studied the notion of bi-bases of a semigroup (Kummoon & Changpas, 2017) and bi-bases of Γ-semigroup (Kummoon & Changpas, 2017).

This is an algebraic structure, generalized the concept of semigroups, called a Γ-semigroup introduced by Sen (1981). The notion of a Γ-semigroup was defined as a generalization of a semigroup by the following definition (Sen & Saha, 1986; Saha, 1987; Saha, 1998).

Definition 1.1. Let $M = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be any two non-empty sets. Then M is said to be a Γ-semigroup if it satisfies the two following conditions:

(1) $x\alpha y \in M$ for all $x, y \in M$ and $\alpha \in \Gamma$;

(2) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Let M be a Γ-semigroup. If A and B are two subsets of M, we shall denote the $A\Gamma B := \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$. We also write $a\Gamma B, A\Gamma b$ and $a\Gamma b$ for $\{a\}\Gamma B, A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively.

Definition 1.2. Let M be a Γ-semigroup and a nonempty subset A of M is called a sub-Γ-semigroup of M if $A\Gamma A \subseteq T$.

The main purpose of this paper is to introduce the concept of bi-bases of an ordered Γ-semigroup and extend some of bi-bases of Γ-semigroup results . Ordered Γ-semigroup was studied by Kehayopula (Kehayopulu, 2010). In 2009, Chinram and Tinpun (2009) studied some properties of bi-ideals and minimal bi-ideals in ordered Γ-semigroups.

Definition 1.3. $(M, \Gamma, \leq)$ is called an ordered Γ-semigroup if (M, Γ) is a Γ-semigroup and $(M, \leq)$ is a partially ordered set such that $a \leq b \Rightarrow a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in M$ and $\gamma \in \Gamma$.

if $(M; \leq)$ is an ordered Γ-semigroup, and K is a sub-Γ-semigroup of M, then $(K; \leq)$ is an ordered Γ-semigroup. For an element a of ordered Γ-semigroup M, define $(a] := \{t \in M \mid t \leq a\}$ and for a subset H of M, define $(H]=\bigcup_{h\in H}(h]$, that is, $(H]=\{t\in M \mid t\le h$ for subset $h\in H\}$, and $H\cup a := H\cup\{a\}$. We observe here that

1. $H\subseteq (H]=\ (H]]$.
2. For any subsets A and B of M with $A\subseteq B$, we have $(A]\subseteq (B]$.
3. For any subsets A and B of M, we have $(A\cup B]=(A]\cup(B]$.
4. For any subsets A and B of M, we have $(A\cap B]\subseteq (A]\cap(B]$.
5. For any subsets a and b of M with $a\le b$, we have $(a\Gamma M]\subseteq(b\Gamma M]$ and $(M\Gamma a]\subseteq(M\Gamma b]$.

Definition 1.4. Let $(M,\Gamma,\le)$ be an ordered Γ-semigroup. A nonempty subset A of M is called a bi-ideal of M if the following hold.

1. $B\Gamma M\Gamma B\subseteq B$
2. If $x\in B$, and $y\in M$, such that $y\le x$, then $y\in B$.

In 2009, lampan give some results which are necessary in ordered bi-ideals of M (lampan, 2009).

Lemma 1.5. For any nonempty subset A of a ordered Γ-semigroup M, $(A\cup A\Gamma A\cup A\Gamma M\Gamma A]$ is the smallest ordered bi-ideal of M containing A. Furthermore, for any $a\in M$,

$$(a)_b=(a\cup a\Gamma a\cup a\Gamma M\Gamma a].$$

Lemma 1.6. Let $\{B_i \mid i\in I\}$ be a family of ordered bi-ideals of M Then $\bigcap_{i\in I}B_i$ is an ordered bi-ideal of M if $\bigcap_{i\in I}B_i\neq\varnothing$.

Main Results

We begin this section with the following definition of bi-bases of an ordered Γ-semigroup.

Definition 2.1. Let M be an ordered Γ-semigroup. A subset B of called a bi-base of M if it satisfies the two following conditions:

1. $M=(B)_b$;
2. if A is a subset of B such that $M=(A)_b$, then $A=B$.

Example 2.2 Let $M=\{a,b,c,d\}$ and $\Gamma=\{\alpha,\beta\}$ where α,β is defined on M with the following Cayley tables:

α	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	c
d	a	c	c	c

β	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	c
d	a	b	c	d

$$\le := \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(d,c),(d,d)\}.$$

In (Chinnadurai & Arulmozhi, 2018), we have shown that $(M,\Gamma,\le)$ is an ordered Γ-semigroup. Consider $B_1^{'}=\{b\}$ and $B_2^{'}=\{d\}$ is not a bi-base of M. But $B_1=\{b,d\}$ is a bi-base of M.

Lemme 2.3 Let B be a bi-base of an ordered Γ-semigroup M. Let $a,b\in B$. If $a\in(b\Gamma b\cup b\Gamma M\Gamma b]$, then $a=b$.

Proof. Assume that $a \in (b\Gamma b \cup b\Gamma M\Gamma b]$, and suppose that $a \neq b$. Let $A := B \setminus \{a\}$. It is clearly seen that $A \subset B$. Since $a \neq b$, $b \in A$, we will show that $(A)_b = M$. Clearly, $(A)_b \subseteq M$.

Next, we show that $M \subseteq (A)_b$. Let $x \in M$. By hypothesis, we have $(B)_b = M$ and so $x \in (B \cup B\Gamma B \cup B\Gamma M\Gamma B]$. Since $x \in (B \cup B\Gamma B \cup B\Gamma M\Gamma B]$, we have $x \leq y$ for some $y \in B \cup B\Gamma B \cup B\Gamma M\Gamma B$. We can consider the three following cases.

Case 1: $y \in B$. There two subcases to consider.

Subcase 1.1: $y \neq a$. Then $y \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $y = a$. By assumption, we have

$$y = a \in (b\Gamma b \cup b\Gamma M\Gamma b] \subseteq (A\Gamma A \cup A\Gamma M\Gamma A] \subseteq (A)_b.$$

Case 2: $y \in B\Gamma B$. Then $y = b_1\gamma b_2$ for some b_1, $b_2 \in B$ and $\gamma \in \Gamma$. There are four subcases to consider.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption, so we have the following:

$$\begin{aligned} y = b_1\gamma b_2 &\in (b\Gamma b \cup b\Gamma M\Gamma b]\Gamma(b\Gamma b \cup b\Gamma M\Gamma b] \\ &\subseteq ((b\Gamma b \cup b\Gamma M\Gamma b)\Gamma(b\Gamma b \cup b\Gamma M\Gamma b)] \\ &= (b\Gamma b\Gamma b\Gamma b \cup b\Gamma b\Gamma b\Gamma M\Gamma b \cup b\Gamma M\Gamma b\Gamma b\Gamma b \cup b\Gamma M\Gamma b\Gamma b\Gamma M\Gamma b] \\ &\subseteq (A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma A\Gamma A \\ &\quad \cup A\Gamma M\Gamma A\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y = b_1\gamma b_2 &\in (B \setminus \{a\})\Gamma(b\Gamma b \cup b\Gamma M\Gamma b] \\ &\subseteq ((B \setminus \{a\})\Gamma(b\Gamma b \cup b\Gamma M\Gamma b)] \\ &= ((B \setminus \{a\})\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma b\Gamma M\Gamma b] \\ &\subseteq (A\Gamma A\Gamma A \cup A\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y = b_1\gamma b_2 &\in (b\Gamma b \cup b\Gamma M\Gamma b]\Gamma(B \setminus \{a\}) \\ &\subseteq ((b\Gamma b \cup b\Gamma M\Gamma b)\Gamma(B \setminus \{a\})] \\ &= (b\Gamma b\Gamma(B \setminus \{a\}) \cup b\Gamma M\Gamma b\Gamma(B \setminus \{a\})] \\ &\subseteq (A\Gamma A\Gamma A \cup A\Gamma M\Gamma A\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$, from $A = B \setminus \{a\}$. Then

$$y = b_1\gamma b_2 \in (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b.$$

Case 3: $y \in B\Gamma M\Gamma B$. Then $y = b_3\gamma_1 m\gamma_2 b_4$ for some b_3, $b_4 \in B$, γ_1, $\gamma_2 \in \Gamma$ and $m \in M$ There are four subcases to consider.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. By assumption, we have

$$\begin{aligned} y = b_3\gamma_1 m\gamma_2 b_4 &\in (b\Gamma b \cup b\Gamma M\Gamma b]\Gamma M\Gamma(b\Gamma b \cup b\Gamma M\Gamma b] \\ &\subseteq ((b\Gamma b \cup b\Gamma M\Gamma b)\Gamma M\Gamma(b\Gamma b \cup b\Gamma M\Gamma b)] \\ &= (b\Gamma b\Gamma M\Gamma b\Gamma b \cup b\Gamma b\Gamma M\Gamma b\Gamma M\Gamma b \cup b\Gamma M\Gamma b\Gamma M\Gamma b\Gamma b \\ &\quad \cup b\Gamma M\Gamma b\Gamma M\Gamma b\Gamma M\Gamma b] \\ &\subseteq (A\Gamma A\Gamma M\Gamma A\Gamma A \cup A\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma \\ &\quad A\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y = b_3\gamma_1 m\gamma_2 b_4 &\in (B \setminus \{a\})\Gamma M\Gamma(b\Gamma b \cup b\Gamma M\Gamma b] \\ &\subseteq ((B \setminus \{a\})\Gamma M\Gamma(b\Gamma b \cup b\Gamma M\Gamma b)] \\ &= ((B \setminus \{a\})\Gamma M\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma M\Gamma b\Gamma M\Gamma b] \\ &\subseteq (A\Gamma M\Gamma A\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y = b_3\gamma_1 m\gamma_2 b_4 &\in (b\Gamma b \cup b\Gamma M\Gamma b]\Gamma M\Gamma(B \setminus \{a\}) \\ &\subseteq ((b\Gamma b \cup b\Gamma M\Gamma b)\Gamma M\Gamma(B \setminus \{a\})] \\ &= (b\Gamma b\Gamma M\Gamma(B \setminus \{a\}) \cup b\Gamma M\Gamma b\Gamma M\Gamma(B \setminus \{a\})] \\ &\subseteq (A\Gamma A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$, hence

$$y = b_3\gamma_1 m\gamma_2 b_4 \in (B \setminus \{a\})\Gamma M\Gamma(B \setminus \{a\}) \subseteq A\Gamma M\Gamma A \subseteq (A)_b.$$

By case 1, 2 and 3 we have $M \subseteq (A)_b$. This implies $(A)_b = M$. This is a contradiction. Therefore, $a = b$.

Lemme 2. 4. Let B be a bi- base of an ordered Γ- semigroup M. Let $a, b, c \in B$. If $a \in (c\Gamma b \cup c\Gamma M\Gamma b]$, then $a = b$ or $a = c$.

Proof. Assume that $a \in (c\Gamma b \cup c\Gamma M\Gamma b]$. Suppose that $a \neq b, a \neq c$ let $A := B \setminus \{a\}$, then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(A)_b = M$. Clearly, $(A)_b \subseteq M$. Let $x \in M$, we need to prove only that $M \subseteq (A)_b$. Since B is a bi-base of M, we have $x \in (B \cup B\Gamma B \cup B\Gamma M\Gamma B]$. Since we consider $x \in (B \cup B\Gamma B \cup B\Gamma M\Gamma B]$, then $x \leq y$ for some $y \in B \cup B\Gamma B \cup B\Gamma M\Gamma B$. We can consider the three following cases.

Case 1: $y \in B$. There are two subcases to consider.

Subcase 1.1: $y \neq a$. Then $y \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $y = a$. By assumption, we have

$$y = a \in (c\Gamma b \cup c\Gamma M\Gamma b] \subseteq (A\Gamma A \cup A\Gamma M\Gamma A] \subseteq (A)_b.$$

Case 2: $y \in B\Gamma B$. Then $y = b_1\gamma b_2$ for some b_1, $b_2 \in B$ and $\gamma \in \Gamma$. There are four subcases to consider.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption, we have

$$\begin{aligned}
y = b_1\gamma b_2 &\in (c\Gamma b \cup c\Gamma M\Gamma b]\Gamma(c\Gamma b \cup c\Gamma M\Gamma b] \\
&\subseteq ((c\Gamma b \cup c\Gamma M\Gamma b)\Gamma(c\Gamma b \cup c\Gamma M\Gamma b)] \\
&= (c\Gamma b\Gamma c\Gamma b \cup c\Gamma b\Gamma c\Gamma M\Gamma b \cup c\Gamma M\Gamma b\Gamma c\Gamma b \cup c\Gamma M\Gamma b\Gamma c\Gamma M\Gamma b] \\
&\subseteq (A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma A\Gamma A \\
&\quad \cup A\Gamma M\Gamma A\Gamma A\Gamma M\Gamma A] \\
&\subseteq (A\Gamma M\Gamma A] \\
&\subseteq (A)_b.
\end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned}
y = b_1\gamma b_2 &\in (B \setminus \{a\})\Gamma(c\Gamma b \cup c\Gamma M\Gamma b] \\
&\subseteq (B \setminus \{a\}]\Gamma(c\Gamma b \cup c\Gamma M\Gamma b] \\
&\subseteq ((B \setminus \{a\})\Gamma(c\Gamma b \cup c\Gamma M\Gamma b)] \\
&= ((B \setminus \{a\})\Gamma c\Gamma b \cup (B \setminus \{a\})\Gamma c\Gamma M\Gamma b] \\
&\subseteq (A\Gamma A\Gamma A \cup A\Gamma A\Gamma M\Gamma A] \\
&\subseteq (A\Gamma M\Gamma A] \\
&\subseteq (A)_b.
\end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned}
y = b_1\gamma b_2 &\in (c\Gamma b \cup c\Gamma M\Gamma b]\Gamma(B \setminus \{a\}) \\
&\subseteq (c\Gamma b \cup c\Gamma M\Gamma b]\Gamma(B \setminus \{a\}] \\
&\subseteq ((c\Gamma b \cup c\Gamma M\Gamma b)\Gamma(B \setminus \{a\})] \\
&= (c\Gamma b\Gamma(B \setminus \{a\}) \cup c\Gamma M\Gamma b\Gamma(B \setminus \{a\})] \\
&\subseteq (A\Gamma A\Gamma A \cup A\Gamma M\Gamma A\Gamma A] \\
&\subseteq (A\Gamma M\Gamma A] \\
&\subseteq (A)_b.
\end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B \setminus \{a\}$, hence

$$y = b_1\gamma b_2 \in (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b.$$

Case 3: $y \in B\Gamma M\Gamma B$. Then $y = b_3\gamma_1 m\gamma_2 b_4$ for some b_3, $b_4 \in B$, γ_1, $\gamma_2 \in \Gamma$ and $m \in M$. There are four subcases to consider.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. By assumption, we have

$$\begin{aligned} y = b_3\gamma_1 m\gamma_2 b_4 &\in (c\Gamma b \cup c\Gamma M\Gamma b]\Gamma M\Gamma(c\Gamma b \cup c\Gamma M\Gamma b] \\ &\subseteq ((c\Gamma b \cup c\Gamma M\Gamma b)\Gamma M\Gamma(c\Gamma b \cup c\Gamma M\Gamma b)] \\ &= (c\Gamma b\Gamma M\Gamma c\Gamma b \cup c\Gamma b\Gamma M\Gamma c\Gamma M\Gamma b \cup c\Gamma M\Gamma b\Gamma M\Gamma c\Gamma b \\ &\quad \cup c\Gamma M\Gamma b\Gamma M\Gamma c\Gamma M\Gamma b] \\ &\subseteq (A\Gamma A\Gamma M\Gamma A\Gamma A \cup A\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma A \\ &\quad \cup A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y = b_3\gamma_1 m\gamma_2 b_4 &\in (B \setminus \{a\})\Gamma M\Gamma(c\Gamma b \cup c\Gamma M\Gamma b] \\ &\subseteq (B \setminus \{a\}]\Gamma(M]\Gamma(c\Gamma b \cup c\Gamma M\Gamma b] \\ &\subseteq ((B \setminus \{a\})\Gamma M\Gamma(c\Gamma b \cup c\Gamma M\Gamma b)] \\ &= ((B \setminus \{a\})\Gamma M\Gamma c\Gamma b \cup (B \setminus \{a\})\Gamma M\Gamma c\Gamma M\Gamma b] \\ &\subseteq (A\Gamma M\Gamma A\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y = b_3\gamma_1 m\gamma_2 b_4 &\in (c\Gamma b \cup c\Gamma M\Gamma b]\Gamma M\Gamma(B \setminus \{a\}) \\ &\subseteq (c\Gamma b \cup c\Gamma M\Gamma b]\Gamma(M]\Gamma(B \setminus \{a\}] \\ &\subseteq ((c\Gamma b \cup c\Gamma M\Gamma b)\Gamma M\Gamma(B \setminus \{a\})] \\ &= (c\Gamma b\Gamma M\Gamma(B \setminus \{a\}) \cup c\Gamma M\Gamma b\Gamma M\Gamma(B \setminus \{a\})] \\ &\subseteq (A\Gamma A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma A] \\ &\subseteq (A\Gamma M\Gamma A] \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$, hence

$$y = b_3\gamma_1 m\gamma_2 b_4 \in (B \setminus \{a\})\Gamma M\Gamma(B \setminus \{a\}) \subseteq A\Gamma M\Gamma A \subseteq (A)_b.$$

By case 1, 2 and 3 we have $M \subseteq (A)_b$. This implies $(A)_b = M$. This is a contradiction. Therefore, $a = b$.

To characterize when a non-empty subset of an ordered Γ-semigroup is a bi-base of the ordered Γ-semigroup, we define the quasi-order defined as follows:

Definition 2.5. Let M be an ordered Γ-semigroup Define a quasi-order on M by, for any $a, b \in M$,

$$a \leq_b b \Leftrightarrow (a)_b \subseteq (b)_b.$$

The following examples show that the order $\leq_b$ defined above is not, in general, a partial order.

Example 2.6. From Example 2.2, we have that $(b)_b \subseteq (d)_b$ (i.e., $b \leq_b d$) and $(d)_b \subseteq (b)_b$ (i.e., $d \leq_b b$) but $b \neq d$. Thus, $\leq_b$ is not a partial order on M.

If A is a bi-base of M, then $(A)_b = M$. Let $x \in M$. Then $x \in (A)_b$ and so $x \in (a)_b$ for some $a \in A$. This implies $(x)_b \subseteq (a)_b$. Hence $x \leq_b a$. Then we can conclude that:

Remark 2.7. A non-empty subset B of an ordered Γ-semigroup $(M, \Gamma, \leq)$. If B is a bi-base of M, then for any $x \in H$ there exists $a \in B$ such that $x \leq_b a$.

Lemma 2.8. Let B be a bi-base of an ordered Γ-semigroup M. If $a, b \in B$ such that $a \neq b$, then neither $a \leq_b b$, nor $b \leq_b a$.

Proof. Assume that $a, b \in B$ such that $a \neq b$. Suppose $a \leq_b b$. Let $A = B \setminus \{a\}$. Then $b \in A$. Let $x \in M$. By Remark 2.7, there exists $c \in B$ such that $x \leq_b c$. We devide two cases to consider. If $c \neq a$, then $c \in A$ thus $(x)_b \subseteq (c)_b \subseteq (A)_b$. Hence $M = (A)_b$, this is a contradiction. If $c = a$, then $x \leq_b b$. Hence $x \in (A)_b$ since $b \in A$. We have $M = (A)_b$, this is a contradiction. In case $b \leq_b a$, we can prove similarly.

Lemma 2.9. Let B be a bi-base of an ordered Γ-semigroup M. Let $a, b, c \in B$ and $\gamma_1, \gamma_2 \in \Gamma$ and $m \in M$.

(1) If $a \in (\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma M \Gamma\{b\gamma_1 c\}]$, then $a = b$ or $a = c$.

(2) If $a \in (\{b\gamma_1 m\gamma_2 c\} \cup \{b\gamma_1 m\gamma_2 c\}\Gamma\{b\gamma_1 m\gamma_2 c\} \cup \{b\gamma_1 m\gamma_2 c\}\Gamma M \Gamma\{b\gamma_1 m\gamma_2 c\}]$, then $a = b$ or $a = c$.

Proof. (1) Assume that $a \in (\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma M \Gamma\{b\gamma_1 c\}]$, and suppose that $a \neq b$ and $a \neq c$. Let $A := B \setminus \{a\}$. Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(B)_b \subseteq (A)_b$, it suffices to show that $B \subseteq (A)_b$. let $x \in B$, if $x \neq a$, that $x \in A$, and so $x \in (A)_b$. If $x = a$, then by assumption we have

$$
\begin{aligned}
x = a &\in (\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma M \Gamma\{b\gamma_1 c\}] \\
&\subseteq (A\Gamma A \cup A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma M\Gamma A\Gamma A] \\
&\subseteq (A\Gamma A \cup A\Gamma M\Gamma A] \\
&\subseteq (A)_b.
\end{aligned}
$$

Thus, $B \subseteq (A)_b$. This implies $(B)_b \subseteq (A)_b$. Since B is a bi-base of M and $M = (B)_b \subseteq (A)_b \subseteq M$, therefore, $S = (A)_b$, this is a contradiction.

(2) Assume that $a \in (\{b\gamma_1 m\gamma_2 c\} \cup \{b\gamma_1 m\gamma_2 c\}\Gamma\{b\gamma_1 m\gamma_2 c\} \cup \{b\gamma_1 m\gamma_2 c\}\Gamma M \Gamma\{b\gamma_1 m\gamma_2 c\}]$, and suppose that $a \neq b$ and $a \neq c$. Let $A := B \setminus \{a\}$. Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(B)_b \subseteq (A)_b$, it suffices to show that $B \subseteq (A)_b$. let $x \in B$. If $x \neq a$, that $x \in A$, and so $x \in (A)_b$. If $x = a$, then by assumption we have

$$
\begin{aligned}
y = a &\in (\{b\gamma_1 m\gamma_2 c\} \cup \{b\gamma_1 m\gamma_2 c\}\Gamma\{b\gamma_1 m\gamma_2 c\} \cup \{b\gamma_1 m\gamma_2 c\}\Gamma M \Gamma\{b\gamma_1 m\gamma_2 c\}] \\
&\subseteq (A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma A\Gamma M\Gamma A \cup A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma M\Gamma A] \\
&\subseteq (A\Gamma M\Gamma A]
\end{aligned}
$$

$\subseteq (A)_b$.

Thus, $B \subseteq (A)_b$. This implies $(B)_b \subseteq (A)_b$. Since B is a bi-base of M and $M = (B)_b \subseteq (A)_b \subseteq M$, therefore, $S = (A)_b$, this is a contradiction.

Lemma 2.10. Let B be a bi-base of an ordered Γ-semigroup M.

(1) For any $a, b, c \in B, \gamma_1 \in \Gamma$, if $a \neq b$ and $a \neq c$, then $a \nleq_b b\gamma_1 c$.

(2) For any $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $m \in M$, if $a \neq b$ and $a \neq c$, then $a \nleq_b b\gamma_2 m\gamma_3 c$.

Proof. (1) For any $a, b, c \in B, \gamma_1 \in \Gamma$, let $a \neq b$ and $a \neq c$. Suppose that $a \leq_b b\gamma_1 c$, we have $a \in (a)_b \subseteq (b\gamma_1 c)_b = (\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma\{b\gamma_1 c\} \cup \{b\gamma_1 c\}\Gamma M\Gamma\{b\gamma_1 c\}]$. By Lemma 2.9 (1), it follows that $a = b$ or $a = c$, this contradicts to the assumption.

(2) For any $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $m \in M$, let $a \neq b$ and $a \neq c$. Suppose that $a \leq_b b\gamma_2 m\gamma_3 c$, we have

$a \in (a)_b \subseteq (b\gamma_2 m\gamma_3 c)_b = (\{b\gamma_2 m\gamma_3 c\} \cup \{b\gamma_2 m\gamma_3 c\}\Gamma\{b\gamma_2 m\gamma_3 c\} \cup \{b\gamma_2 m\gamma_3 c\}\Gamma M\Gamma\{b\gamma_2 m\gamma_3 c\}]$.

By Lemma 2.9 (2), it follows that $a = b$ or $a = c$, this contradicts to the assumption.

The following theorem characterizes when a non-empty subset of an ordered Γ-semigroup M is a bi-base of M.

Theorem 2.11. A non-empty subset B of an ordered Γ-semigroup M is a bi-base of M if and only if B satisfies the following conditions:

(1) For any $x \in M$,

(1.1) there exists $b \in B$ such that $x \leq_b b$; or

(1.2) there exists $b_1, b_2 \in B$ and $\gamma \in \Gamma$ such that $x \leq_b b_1\gamma b_2$; or

(1.3) there exists $b_3, b_4 \in B, m \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x \leq_b b_3\gamma_1 m\gamma_2 b_4$.

(2) For any $a, b, c \in B, \gamma_1 \in \Gamma$, if $a \neq b$ and $a \neq c$, then $a \nleq_b b\gamma_1 c$.

(3) For any $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $m \in M$, if $a \neq b$ and $a \neq c$, then $a \nleq_b b\gamma_2 m\gamma_3 c$.

Proof. Assume first B is a bi-base of M, then $M = (B)_b$. To show that (1) holds, Let $x \in M$, then $x \in (B \cup B\Gamma B \cup B\Gamma M\Gamma B]$. Since $x \in (B \cup B\Gamma B \cup B\Gamma M\Gamma B]$, We have $x \leq y$ for some $y \in B \cup B\Gamma B \cup B\Gamma M\Gamma B$. We consider three cases:

Case 1: $y \in B$. Then $y = b$ for some $b \in B$. This implies $(y)_b \subseteq (b)_b$. Hence, $y \leq_b b$. Since $x \leq y$ for some $y \in (b)_b$, we have $x \in (b)_b$. We will show $(x)_b \subseteq (b)_b$. Consider

$$\begin{aligned} x \cup x\Gamma x \cup x\Gamma M\Gamma x &\subseteq (b)_b \cup (b)_b\Gamma(b)_b \cup (b)_b\Gamma M\Gamma(b)_b \\ &\subseteq (b \cup b\Gamma b \cup b\Gamma M\Gamma b]. \end{aligned}$$

Then we have $(x \cup x\Gamma x \cup x\Gamma M\Gamma x] \subseteq (b \cup b\Gamma b \cup b\Gamma M\Gamma b]]$. This implies $(x)_b \subseteq (b)_b$. Hence, $x \leq_b b$.

Case 2: $y \in B\Gamma B$. Then $y = b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$. This implies $(y)_b \subseteq (b_1\gamma b_2)_b$. Hence, $y \leq_b b_1\gamma b_2$. Since $x \leq y$ for some $y \in (b_1\gamma b_2)_b$, we have $x \in (b_1\gamma b_2)_b$. We will show that $(x)_b \subseteq (b_1\gamma b_2)_b$. Consider

$$x \cup x\Gamma x \cup x\Gamma M\Gamma x \subseteq (b_1\gamma b_2)_b \cup (b_1\gamma b_2)_b\Gamma(b_1\gamma b_2)_b \cup (b_1\gamma b_2)_b\Gamma M\Gamma(b_1\gamma b_2)_b$$
$$\subseteq (\{b_1\gamma b_2\} \cup \{b_1\gamma b_2\}\Gamma\{b_1\gamma b_2\} \cup \{b_1\gamma b_2\}\Gamma M\Gamma\{b_1\gamma b_2\}].$$

Then we have $(x \cup x\Gamma x \cup x\Gamma M\Gamma x] \subseteq ((\{b_1\gamma b_2\} \cup \{b_1\gamma b_2\}\Gamma\{b_1\gamma b_2\} \cup \{b_1\gamma b_2\}\Gamma M\Gamma\{b_1\gamma b_2\}]]$. This implies $(x)_b \subseteq (b_1\gamma b_2)_b$. Hence, $x \leq_b b_1\gamma b_2$.

Case 3 : $y \in B\Gamma M\Gamma B$. Then $y = b_3\gamma_1 m\gamma_2 b_4$ for some $b_3, b_4 \in B$ and $\gamma_1, \gamma_2 \in \Gamma$. This implies $(y)_b \subseteq (b_3\gamma_1 m\gamma_2 b_4)_b$. Hence, $y \leq_b (b_3\gamma_1 m\gamma_2 b_4)_b$. Since $x \leq y$ for some $y \in (b_3\gamma_1 m\gamma_2 b_4)_b$, we have $x \in (b_3\gamma_1 m\gamma_2 b_4)_b$. We will show that $(x)_b \subseteq (b_3\gamma_1 m\gamma_2 b_4)_b$. Consider $x \cup x\Gamma x \cup x\Gamma M\Gamma x \subseteq (b_3\gamma_1 m\gamma_2 b_4)_b \cup (b_3\gamma_1 m\gamma_2 b_4)_b\Gamma(b_3\gamma_1 m\gamma_2 b_4)_b \cup (b_3\gamma_1 m\gamma_2 b_4)_b\Gamma M\Gamma(b_3\gamma_1 m\gamma_2 b_4)_b \subseteq (\{b_3\gamma_1 m\gamma_2 b_4\} \cup \{b_3\gamma_1 m\gamma_2 b_4\}\Gamma\{b_3\gamma_1 m\gamma_2 b_4\} \cup \{b_3\gamma_1 m\gamma_2 b_4\}\Gamma M\Gamma\{b_3\gamma_1 m\gamma_2 b_4\}]$. Then we have $(x \cup x\Gamma x \cup x\Gamma M\Gamma x] \subseteq ((\{b_3\gamma_1 m\gamma_2 b_4\} \cup \{b_3\gamma_1 m\gamma_2 b_4\}\Gamma\{b_3\gamma_1 m\gamma_2 b_4\} \cup \{b_3\gamma_1 m\gamma_2 b_4\}\Gamma M\Gamma\{b_3\gamma_1 m\gamma_2 b_4\}]]$. This implies $(x)_b \subseteq (b_3\gamma_1 m\gamma_2 b_4)_b$. Hence, $x \leq_b b_3\gamma_1 m\gamma_2 b_4$. The validity of (2) and (3) follows, respectively, from Lemma 2.10 (1) and Lemma 2.10 (2)

Conversely, assume that the conditions (1), (2) and (3) are hold. We will show that B is a bi-base of M. To show that $M = (B)_b$. Clearly, $(B)_b \subseteq M$. By (1) $M \subseteq (B)_b$ and $M = (B)_b$. It remains to show that B is a minimal subset of M, with the property: $M = (B)_b$. Suppose that $M = (A)_b$ for some $A \subset B$. Since $A \subset B$, there exists $b \in B \setminus A$. Since $b \in B \subseteq M = (A)_b$ and $b \notin A$, it follows that $b \in (A\Gamma A \cup A\Gamma M\Gamma A]$. Since $b \in (A\Gamma A \cup A\Gamma M\Gamma A]$, we have $b \leq y$ for some $y \in A\Gamma A \cup A\Gamma M\Gamma A$. There are two cases to consider:

Case 1: $y \in A\Gamma A$. Then $y = a_1\gamma_1 a_2$ for some $a_1, a_2 \in A$ and $\gamma_1 \in \Gamma$. We have $a_1, a_2 \in B$. Since $b \notin A$, so $b \neq a_1$ and $b \neq a_2$. Since $y = a_1\gamma_1 a_2$, $(y)_b \subseteq (a_1\gamma_1 a_2)_b$. Hence, $y \leq_b a_1\gamma_1 a_2$. Since $b \leq y$ for some $y \in (a_1\gamma_1 a_2)_b$, we have $b \in (a_1\gamma_1 a_2)_b$. We will show that $(b)_b \subseteq (a_1\gamma_1 a_2)_b$. Consider

$$b \cup b\Gamma b \cup b\Gamma M\Gamma b \subseteq (a_1\gamma_1 a_2)_b \cup (a_1\gamma_1 a_2)_b\Gamma(a_1\gamma_1 a_2)_b \cup (a_1\gamma_1 a_2)_b\Gamma M\Gamma(a_1\gamma_1 a_2)_b$$
$$\subseteq (\{a_1\gamma_1 a_2\} \cup \{a_1\gamma_1 a_2\}\Gamma\{a_1\gamma_1 a_2\} \cup \{a_1\gamma_1 a_2\}\Gamma M\Gamma\{a_1\gamma_1 a_2\}].$$

Then we have $(b \cup b\Gamma b \cup b\Gamma M\Gamma b] \subseteq ((\{a_1\gamma_1 a_2\} \cup \{a_1\gamma_1 a_2\}\Gamma\{a_1\gamma_1 a_2\} \cup \{a_1\gamma_1 a_2\}\Gamma M\Gamma\{a_1\gamma_1 a_2\}]]$. This implies $(b)_b \subseteq (a_1\gamma_1 a_2)_b$ Hence, $b \leq_b a_1\gamma_1 a_2$. This contradicts to (2).

Case 2: $y \in A\Gamma M\Gamma A$. Then $y = a_3\gamma_2 m\gamma_3 a_4$ for some $a_3, a_4 \in A$, $\gamma_2, \gamma_3 \in \Gamma$ and $m \in M$. Since $b \notin A$, we have $b \neq a_3$ and $b \neq a_4$. Since $A \subset B, a_3, a_4 \in B$. Since $y = a_3\gamma_2 m\gamma_3 a_4$, so $(y)_b \subseteq (a_3\gamma_2 m\gamma_3 a_4)_b$. Hence, $y \leq_b a_3\gamma_2 m\gamma_3 a_4$. Since $b \leq y$ for some $y \in (a_3\gamma_2 m\gamma_3 a_4)_b$, we have $b \in (a_3\gamma_2 m\gamma_3 a_4)_b$. We will show that $(b)_b \subseteq (a_3\gamma_2 m\gamma_3 a_4)_b$. Consider

$b \cup b\Gamma b \cup b\Gamma M\Gamma b \subseteq (a_3\gamma_1 m\gamma_2 a_4)_b \cup (a_3\gamma_1 m\gamma_2 a_4)_b\Gamma(a_3\gamma_1 m\gamma_2 a_4)_b \cup (a_3\gamma_1 m\gamma_2 a_4)_b\Gamma M\Gamma(a_3\gamma_1$ $m\gamma_2 a_4)_b \subseteq (\{a_3\gamma_1 m\gamma_2 a_4\} \cup \{a_3\gamma_1 m\gamma_2 a_4\}\Gamma\{a_3\gamma_1 m\gamma_2 a_4\} \cup \{a_3\gamma_1 m\gamma_2 a_4\}\Gamma M\Gamma\{a_3\gamma_1 m\gamma_2 a_4\}]$.
Thenwe have

$(b \cup b\Gamma b \cup b\Gamma M\Gamma b] \subseteq$ $(\{a_3\gamma_1 m\gamma_2 a_4\} \cup \{a_3\gamma_1 m\gamma_2 a_4\}\Gamma\{a_3\gamma_1 m\gamma_2 a_4\} \cup \{a_3\gamma_1 m\gamma_2 a_4\}$ $\Gamma M\Gamma$ $\{a_3\gamma_1 m\gamma_2 a_4\}]]$ This implies $(b)_b \subseteq (a_3\gamma_1 m\gamma_2 a_4)_b$. Hence, $x \leq_b$ $a_3\gamma_1 m\gamma_2 a_4$. This contradicts to (3) therefore, B is a bi-base of M as required, and the proof is completed.

Theorem 2.11. Let B be a bi-base of an ordered Γ- semigroup M. Then B is a sub-Γ- semigroup of M if and only if for any $a, b \in B$ and $\beta \in \Gamma, a\beta b = a$ or $a\beta b = b$.

Proof. Let $a, b \in B$ and $\beta \in \Gamma$. If B is a sub- Γ- semigroup of M, then $a\beta b \in B$. Since $a\beta b \in (a\Gamma b \cup a\Gamma M\Gamma b]$, it follows by Lemma 2.4 that $a\beta b = a$ or $a\beta b = b$. The opposite direction is clear.

References

Changphas, T., & Sammaprab, P. (2014). On Two sided bases of an ordered semigroup. *Quasi-group and Related Systems*, *22*(1), 59-66. Retrieved from http://www.quasigroups.eu

Chinnadurai, V., & Arulmozhi, K. (2018). Characterization of bipolar fuzzy ideals in ordered gamma semigroups. *Journal of The International Mathematical Virtual Institute*, *8*, 141-156. Retrieved from https://www.imvibl.org/journal.htm

Chinram, R., & Tinpun, K. (2009). A note on minimal bi-ideals in ordered Γ-semigroups. *International Mathematical Forum*, *4*(1), 1-5. Retrieved from http://www.m-hikari.com/aims.html

Fabrici, I. (1975). Two-sided bases of semigroups. *Matematick´y ˇCasopis*, *25*(2), 173-178. Retrieved from http://dml.cz/dmlcz/126947

Iampan, A. (2009). Characterizing Ordered Bi- Idealsin Ordered Γ- Semigroups. *Iranian Journal of Mathematical Sciences and Informatics*, *4*(1), 17-25. Retrieved from http://dx.doi.org/10.7508/ijmsi.2009.01.002

Kehayopulu, N. (2010), On ordered Γ-semigroups. *Scientiae Mathematicae Japonicae*, *23*, 37-43.

Kummoon, P., & Changphas, T. (2017). Bi-Bases of Γ-semigroup. *Thai Journal of Mathematics*, *75-86*. Retrieved from http://thaijmath.in.cmu.ac.th

Kummoon, P., & Changphas, T. (2017). On bi-base of semigroup. *Quasigroup and Related Systems*, *25*(1), 87-94. Retrieved from http://www.quasigroups.eu

Sen, M.K. (1981). On Γ- semigroups. *Proceedings of International Conference on Algebra and It's Applications*, New York, 301-308.

Sen, M. K., & Saha, N. K. (1986). On Γ-semigroup I. *Bulletin of Calcutta Mathematical Society*, *78*, 180-186.

Saha, N. K. (1987). On Γ-semigroup II. *Bulletin of Calcutta Mathematical Society*, *79*, 331-335.

Saha, N. K. (1998). On Γ-semigroup III. *Bulletin of Calcutta Mathematical Society*, *80*, 1-13.