ผลเฉลยโซลิตอนของสมการ KORTEWEG-DE VRIES แบบปรับปรุงโดยวิธีการกระจาย SINE-GORDON

SOLITON SOLUTION TO THE MODIFIED KORTEWEG-DE VRIES EQUATION USING SINE-GORDON EXPANSION METHOD

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บทคัดย่อ

ผลเฉลยโซลิตอนของสมการ sine-Gordon จะถูกเขียนในรูปของฟังก์ชันไฮเปอร์โบลิก ซึ่งจะมีรูปของโซลิตอนทั้งสองแบบ ทำให้สามารถใช้เป็นผลเฉลยของโซลิตอนของสมการไม่เชิงเส้นในรูปอื่นๆ ที่จะหาผลเฉลยโซลิตอน ดังนั้นแนวคิดนี้จะนำมาใช้หาผลเฉลยโซลิตอนของสมการ Korteweg-de Vries ที่มีเทอมไม่เชิงเส้น 2 ตัว โดยจะเขียนในรูปของการกระจายตัวของฟังก์ชันไฮเปอร์โบลิก และทำการหาสัมประสิทธิ์การกระจายตัวตามเงื่อนไขที่เหมาะสมสำหรับผลเฉลยของสมการ นอกจากนี้จะแสดงผลการวิวัฒน์ตามเวลาของผลเฉลยดังกล่าว รวมถึงแสดงผลการชนกันระหว่าง โซลิตอนสองตัวที่ได้จากคำนวณ

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ABSTRACT

The soliton solution of sine-Gordon equation will be determined. These solutions are written in terms of hyperbolic functions which are expressed in two types of the solitons. It can be used for the soliton solutions for the others nonlinear equations for two possible types of solitons. For this idea, we will take the soliton solution of the sine-Gordon equation to obtain the soliton solution of the Korteweg-de Vries equation with two nonlinear terms. The solution will be written in terms of the series of hyperbolic functions. The coefficients of this series will be determined to

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with the suitable condition. Time evolution of the solution will be presented as well as the two solitons collision.

Keywords: soliton, symbolic computing, sine-Gordon expansion

1. Introduction

Solitary wave [1, 2] is the phenomena that wave travels on the continuum media without changing its amplitude. This result was first described mathematically by Korteweg and de Vries [3, 4] which is known as KdV equation,

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

where u is described as the perturbation on media and subscripts x and t are partial derivatives with respect to spatial and temporal coordinates, respectively. The key idea of this phenomenon is the balance between dispersion and nonlinear effects which are the 3rd and 2nd terms on the left hand-side of eq. (1), respectively. However, eq. (1) can also be derived in the ion-acoustic waves in plasmas [5, 6]. To obtain the soliton solution of eq. (1), there are various methods to calculate such as the traveling wave solution [5], Lax-pair method [7] and inverse scattering method [8, 9], tanh method [10], G'/G-expansion method [11] sine-Gordon expansion (sGE) method [12, 13] and Exp-function method [14]. The advantage of the sGE is the combination between pulse and kink solitons in the series solution. We can definitely obtain soliton solution with various possible setting. The modified KdV which will be solved via sGE is KdV with two nonlinear terms [13, 15, 16], the dimensionless form can be written as

$$u_t + uu_x + u^2u_x + u_{xxx} = 0 \tag{2}$$

This equation is also derived in nonlinear optics, plasmas, fluid and some continuum media. The next section, we will find the sine-Gordon solution in order to use as the ansatz solution for the solution of eq. (2)

2. Sine-Gordon expansion method

Consider the sine-Gordon equation,

$$u_{xx} - u_{tt} = m^2 \sin u \tag{3}$$

where m is a constant. This equation is used to derived in many applications of physics [17]. The traveling wave solution can be determined with a new parameter,

$$\xi = \mu(x - ct)$$

in which c is the wave speed and ξ and μ are a constant. The equation will be rewritten as

$$\frac{d^2u}{d\xi^2} = \frac{m^2}{\mu^2(1-c^2)} \sin u.$$

Integrate both side with respect to u,

$$\left(\frac{u_\xi}{2}\right)^2 = \frac{m^2}{\mu^2(1-c^2)} \sin^2\left(\frac{u}{2}\right) + d \tag{4}$$

where d is the integration constant. To find the soliton solution, the initial conditions will be found as $u(0) = 0, u_\xi(0) = 0$ and found that $d = 0$. The final step is to integrate eq. (4) with respect to ξ and it will be useful to redefine $z = \frac{u}{2}$

$$\sin z = \operatorname{sech} a(\xi - \xi_0) \tag{5}$$

and

$$\cos z = \tanh a(\xi - \xi_0) \tag{6}$$

where $a^2 = \frac{m^2}{\mu^2(1-c^2)}$. This implies that the trigonometric functions can be expressed in terms of hyperbolic functions which are the forms of soliton solution. The nonlinear evolution equation can be written as, in general,

$$P_N\left(u, u_t, \dots, \frac{\partial^l u}{\partial x^i \partial t^{l-i}}\right) = 0 \tag{7}$$

The solution of eq. (7) can be expressed as a series solution,

$$u(\xi) = \sum_{k=0}^{n} \tanh^{k-1}\xi \ \ (B_k \operatorname{sech}(\xi) + A_k \tanh(\xi)) + A_0 \tag{8}$$

or be written in terms of eqs. (5) and (6)

$$u(z) = \sum_{k=0}^{n} \cos^{k-1} z \ \ (B_k \sin(z) + A_k \cos(\xi)) + A_0 = P_n(\sin z, \cos z) \tag{9}$$

We will use eqs (8)-(9) as the ansatz solution for eq. (2)

3. Soliton solution to the modified KdV equation

From eq. (9), we will first determine what n will be used for finding the solution. The polynomial degree function, D , is introduced in which returning the degree of the polynomial, such that

$$D(\sin^2 z + 4\cos z) = 2$$
$$D(u(\xi)) = n, \qquad D\big(u^m(\xi)\big) = m \times n$$

and

$$D\left(\frac{d^m}{d\xi^m} u(\xi)\right) = m + n$$

Eq. (2) with the traveling wave solution, $\xi = \mu(x - ct)$,

$$-cu_\xi + uu_\xi + u^2 u_\xi + \mu^2 u_{\xi\xi\xi} = 0 \tag{10}$$

The value of n can determine via comparing on the terms of dispersion and the highest degree for the nonlinear term,

$$D\big(u_{\xi\xi\xi}\big) = n + 3$$
$$D\big(u^2 u_{\xi\xi}\big) = \frac{1}{3} D\left(\frac{d}{d\xi} u^3\right) = 3n + 1$$

Therefore, we have $n = 1$ or the ansatz will be written by

$$u(z) = B_1 \sin z + A_1 \cos z + A_0 \tag{11}$$

where

$$\frac{dz}{d\xi} = \sin z \tag{12}$$

Substitute (11) and (12) into (10)

$$A_0A_1 - A_0^2A_1 - A_1B_1^2 + cA_1 + DA_0A_1 + DA_0^2A_1 - DA_1^3 + 4DA_1B_1^2 - cDA_1 + 2FA_1B_1 + 2FA_1B_1 + 4FA_0A_1B_1 + GA_1^2 + 2GA_0A_1^2 - GB_1^2 - 2GA_0B_1^2 + 3HA_1^2B_1 - HB_1^3 + JA_1^3 - 3JA_1B_1^2 - MA_1^2 - 2MA_0A_1^2 + MB_1^2 + 2MA_0B_1^2 + PA_0B_1 + PA_0^2B_1 - 2PA_1^2B_1 + PB_1^3 - cPB_1 - QA_1B_1 - 2QA_0A_1B_1 + 2\mu^2A_1 - 8D\mu^2A_1 + 6\mu^2HB_1 + 6\mu^2JA_1 - 5\mu^2PB_1 = 0 \tag{13}$$

where

$D = \cos^2 z, F = \sin z * \cos^2 z, G = \cos^3 z, H = \sin z * \cos^3 z, J = \cos^4 z, M = \cos z$

$P = \sin z * \cos z$ and $Q = \sin z$, To determine all unknown, we match all variables in eq. (13),

All constants: $-A_0A_1 - A_0^2A_1 - A_1B_1^2 - cA_1 + 2\mu^2A_1$

Coefficient M: $-A_1^2 - 2A_0A_1^2 + B_1^2 - 2A_0B_1^2$

Coefficient Q: $-A_1B_1 - 2A_0A_1B_1$

Coefficient P: $A_0B_1 + A_0^2B_1 - 2A_1^2B_1 + B_1^3 - cB_1 - 5\mu^2B_1$

Coefficient D: $A_0A_1 + A_0^2A_1 - A_1^3 + 4A_1B_1^2 - cA_1 - 8\mu^2A_1$

Coefficient F: $2A_1B_1 + 4A_0A_1B_1$

Coefficient G: $A_1^2 + 2A_0A_1^2 - B_1^2 - 2A_0B_1^2$

Coefficient H: $3A_1^2B_1 - B_1^3 + 6\mu^2B_1$

Coefficient J: $A_1^3 - 3A_1B_1^2 + 6\mu^2A_1$

Coefficient K: 0

To obtain the kink solution, we will set $B_1 = 0$, all parameters are

$$A_0 = -\frac{1}{2}, c = -\frac{1 + 8\mu^2}{4}, A_1 = 0, A_1 = \pm i\mu\sqrt{6}$$

therefore, we have

$$\xi = \mu\left(x + \frac{(1 + 8\mu^2)t}{4}\right)$$

$$u(\xi) = -\frac{1}{2} + i\mu\sqrt{6}\tanh(\xi) \tag{14}$$

The time evolution of eq. (14) can be shown in Figure 1.

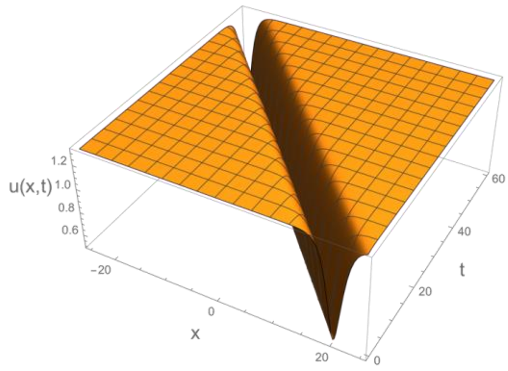

Figure 1. shows the time evolution with $\mu = 0.5$

Another form of the soliton solution is known as the pulse form, this can be set as $A_1 = 0$. All parameters are

$$A_0 = -\frac{1}{2}, c = \frac{-1+4\mu^2}{4}, B_1 = \mu\sqrt{6}$$

The solution will be written as

$$\xi = \mu\left(x + \frac{(-1+4\mu^2)t}{4}\right)$$

$$u(\xi) = -\frac{1}{2} + \mu\sqrt{6}\,\mathrm{sech}(\xi) \quad (14)$$

The time evolution can be shown in Figure. 2

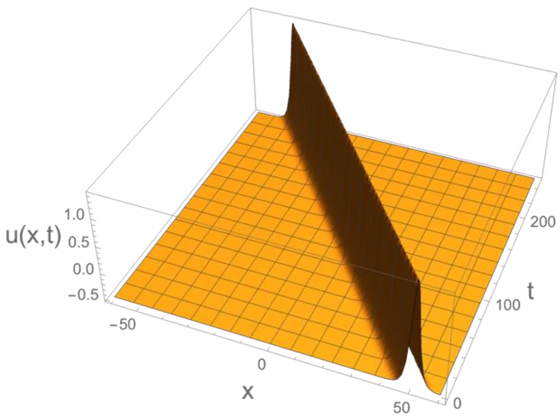

Figure 2. shows the time evolution with $\mu = 0.79$

The head-on collision, where we start with two solutions in the opposite direction, will be shown in Figure 3.

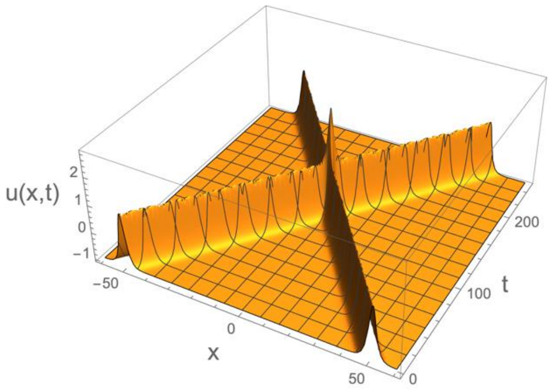

Figure 3. shows the head-on collision with both have the same $\mu = 0.79$

4. Conclusion

Sine-Gordon expansion method can be used for finding the soliton solutions which depends on a number of terms in the series solution. Both pulse and kink shapes are also found with this method. However, the stable solution will be next investigated via the time evolution where these solutions are chosen as the initial condition.

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