KINK WAVE SOLUTIONS FOR THE (1+1)-DIMENSIONAL NONLINEAR EVOLUTION EQUATION BY THE SIMPLE METHOD WITH THE BERNOULLI EQUATION

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ABSTRACT

The main objective of this investigation is to fully solve the nonlinear partial differential equations (1+1)-dimensional Phi-Four and (1+1)-dimensional modified Korteweg-De Vries. Then, with the assistance of the Bernoulli equation, the simple method (SE) will solve these solutions. The solutions are in the form of generalized exponential functions. The effect of the arising graphs of both equations is in the form of kink waves, which are presented in three-dimensional graphs and contour graphs using suitable parameter values in 2 cases.

Keywords: King waves; the simple method with the Bernoulli equation; the (1+1)-dimensional Phi-Four equation; the (1+1)-dimensional modified Korteweg-De Vries equation

1. INTRODUCTION

Numerous significant dynamic processes and phenomena in mathematical physics, fluid mechanics, mechanical engineering, chemistry physics, fluid dynamics, plasma physics, optical fibers, and other engineering domains are described by nonlinear partial differential equation (nPDEs). [1]. In order to precisely understand the qualitative aspects of several occurrences and processes in a variety of natural science domains has always depended on exact or numerical solutions. Many powerful and successful methodologies have been created to deal with the NLEEs, such as the simple method [2-3], the exponential rational function method [4], the tanh-coth method [5], the sin-cosine method [6], the reproducing kernel algorithm method [7-8], the Riccati-Bernoulli sub-ODE method [9], the $\left(G'/G^2\right)$-expansion method [10], the unified method [11]

The Phi-Four equation, which can be considered a particular form of the Klein-Gordon equation that designs the phenomenon in interaction between kink and anti-kink solitary waves in particle physics [12], has the following form:

$$u_{tt} - \alpha u_{xx} - u + u^3 = 0,$$

Where $u = u(x,t)$ and α an arbitrary constant.

The modified Korteweg-De Vries equation has the following form:

$$v_t - v^2 v_x + \delta v_{xxx} = 0,$$

Where $v = v(x,t)$ and δ is a nonzero constant, The expression v_t explain the time evolution of propagating of the wave in one direction. Two opponent effects are also included in this equation: depiction of nonlinearity as $v^2 v_x$ that accounts in order to seep the wave, and v_{xxx} that presented for the spreading of the wave [13].

In this article, the basic method (SE) with the Bernoulli equation is used to solve the (1+1)-dimensional Phi-Four equation and the (1+1)-dimensional modified Korteweg-De Vries equation. We derive the exact solution to these equations in terms of the exponential functions. Furthermore, we show the wave effects in the form of kink waves in a three-dimensional graph and a contour graph.

2. ALGORITHM OF SE METHOD WITH THE BERNOULLI EQUATION

The following is a summary of the fundamental steps of the SE method using the Bernoulli equation [2-3] as follows:

Step 1. Let there be a nonlinear partial differential equation (NLPDE), say, in two independent variables x and t, given by:

$$P\left(u, u_t, u_x, u_{xx}, u_{xt}, \ldots\right) = 0, \quad (1)$$

where P is an exponential function in general of $u(x,t)$ and its arguments; the subscripts denote the partial derivatives. Let, think about integrating the independent variables x and t into one variable, ξ. We suppose that

$$u(x,t)=u(\xi),\ \xi=x-\omega t. \tag{2}$$

Traveling wave transformation Eq. (2) allows us to reduce Eq. (1) to the ordinary differential equation (ODE) that follows:

$$\mathrm{O}(u,u',u'',u''',...)=0, \tag{3}$$

where O is a polynomial in $u(\xi)$ and its total derivatives, where $u'(\xi)=\frac{du}{d\xi}$, $u''(\xi)=\frac{d^2u}{d\xi^2}$, and so on.

Step 2. Suppose that the solution of Eq. (3) is in the following form:

$$u(\xi)=\sum_{i=0}^{N}a_i\kappa^i(\xi)$$

where $a_i (i=0,1,2,...,N)$ are constants that need to be determined such that $a_N \neq 0$. The ODEs, or simple equations, are satisfied by the function $\kappa(\xi)$. This study will make use of the Bernoulli equation, a well-known nODE. It is possible to express their solutions as simple functions. The Bernoulli formula,

$$\kappa'(\xi)=\beta\kappa(\xi)+\eta\kappa^2(\xi). \tag{4}$$

Where β and η are non-zero constants.

Step 3. The derivative of the highest positions and highest nonlinear terms in Eq. (3) could be balanced to obtain the balance number N.

Step 4. We obtain the general answers to Eq. (4) as follows:

Case I: if $\beta>0$ and $\eta<0$, we get

$$\kappa(\xi)=\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}, \tag{5}$$

Case II: if $\beta<0$ and $\eta>0$, we get

$$\kappa(\xi)=-\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}, \tag{6}$$

where ξ_0 is the constant of the integration.

3. RESULTS

Next, using the SE methodology with the Bernoulli explained above, we want to solve the (1+1)-dimensional Phi-Four equation and the (1+1)-dimensional modified Korteweg-De Vries equation as follows.

3.1 Results of the Phi-Four equation

The (1+1)-dimensional Phi-Four equation is

$$u_{tt} - \alpha u_{xx} - u + u^3 = 0, \tag{7}$$

where $u = u(x,t)$ and α an arbitrary constant. Using $U(\xi) = u(x,t)$ and the traveling wave variable $\xi = x - \omega t,$ we will transform it to an ODE Eq. (7) produces the following result when the transformation is applied:

$$(\omega^2 - \alpha)U'' - U + U^3 = 0. \tag{8}$$

The highest nonlinear terms, U^3, and the highest-order derivative terms, U'', were then balanced in Eq. (8). Then $N = 1$. We have the solution to Eq. (8) as follows:

$$U(\xi) = \sum_{i=0}^{1} a_i \kappa^i(\xi) = a_0 + a_1 \kappa, \tag{9}$$

where κ satisfies Eq. (4). Therefore, Therefore, the expressions for U'' and U^3 are expressed as:

$$\begin{aligned} U'' &= a_1\beta^2\kappa + 3a_1\beta\eta\kappa^2 + 2a_1\eta^2\kappa^3, \\ U^3 &= a_0^3 + 3a_0^2a_1\kappa + 3a_0a_1^2\kappa^2 + a_1^3\kappa^3. \end{aligned} \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8), then calculating the coefficient's equation of κ^i to zero, where $i = 0,1,2,3,$ yields

$$\kappa^0(\xi): \; a_0^3 - a_0 = 0, \tag{11}$$

$$\kappa^1(\xi): (\omega^2 - \alpha)a_1\beta^2 - a_1 + 3a_0^2a_1 = 0, \tag{12}$$

$$\kappa^2(\xi): 3(\omega^2 - \alpha)a_1\beta\eta + 3a_0a_1^2 = 0, \tag{13}$$

$$\kappa^3(\xi): 2(\omega^2 - \alpha)a_1\eta^2 + a_1^3 = 0. \tag{14}$$

Solving Eq. (11)-(14), we obtain

$$a_0 = \pm 1,\ a_1 = \pm \frac{2\eta}{\beta} \text{ and } \omega = \sqrt{\frac{\alpha\beta - 2}{\beta^2}}. \tag{15}$$

The following two precise solutions can be obtained by substituting Eq. (15) into Eq. (9). Then, we utilize the general solutions to the Bernoulli Eqs. (5) and (6). We get four exact solutions of (7) written in terms of the exponential function.

Case I: if $\beta > 0$ and $\eta < 0$, we get

$$u_1(x,t) = 1 + 2\eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}\right),$$

$$u_2(x,t) = -1 - 2\eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}\right),$$

where $\xi = x - \omega t$ and ξ_0 is a constant of the integration.

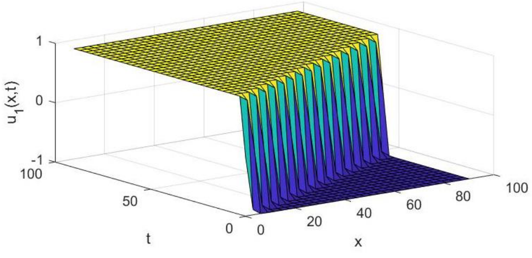
Case II: if $\beta < 0$ and $\eta > 0$, we get

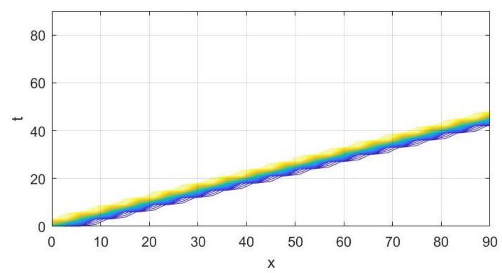
$$u_3(x,t) = 1 - 2\eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}\right),$$

$$u_4(x,t) = -1 + 2\eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}\right),$$

where $\xi = x - \omega t$ and ξ_0 is a constant of the integration.

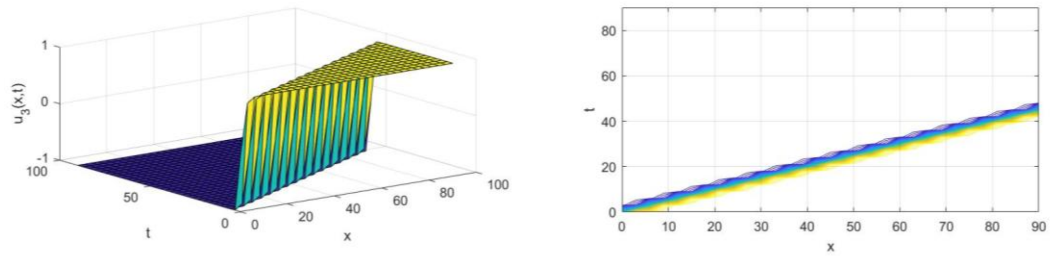
We demonstrate the three-dimensional and contour plot representations of some of the exact solutions, $u_1(x,t)$ in **Case I** and $u_3(x,t)$ in **Case II**, in Figures 1 and 2. We set $\beta = \pm 1,\ \eta = \mp 1,\ \delta = 6,\ \omega = 2$ and $0 \le x, t \le 90$, we obtain the plots of the selected exact traveling wave solution, which represents the kink wave solutions.





$$u_1(x,t) = 1 + 2\eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}\right)$$

Figure 1 Kink wave solution for $u_1(x,t)$ with $\beta = 1, \eta = -1, \alpha = 6$ and $0 \leq x,t \leq 90$



$$u_3(x,t) = 1 - 2\eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}\right)$$

Figure 2 Kink wave solution for $u_3(x,t)$ with $\beta = -1, \eta = 1, \alpha = 6$ and $0 \leq x,t \leq 90$

3.2 Results of the modified Korteweg-De Vries equation

The (1+1)-dimensional modified Korteweg-De Vries equation is

$$v_t - v^2 v_x + \delta v_{xxx} = 0, \tag{16}$$

where $v = v(x,t)$ and δ is a nonzero constant. Using $V(\xi) = v(x,t)$ and the traveling wave variable $\xi = x - \omega t$, we will transform it to an ODE. Eq. (16) produces the following result when the transformation is applied:

$$-\omega V' - V^2 V' + \delta V''' = 0 \tag{17}$$

The highest nonlinear terms, $V'V^2$, and the highest-order derivative terms, V''', were then balanced in Eq. (17). Then $N = 1$. We have the solution to Eq. (17) as follows:

$$V(\xi) = \sum_{i=0}^{1} a_i \kappa^i(\xi) = a_0 + a_1\kappa,$$

where κ satisfies Eq. (5). Therefore, Therefore, the expressions for V', V''' and V^2 are expressed as:

$$\begin{aligned} V' &= a_1\beta\kappa + a_1\eta\kappa^2, \\ V'' &= \beta^2 a_1\kappa + 3\beta\eta a_1\kappa^2 + 2a_1\eta^2\kappa^3, \\ V^2 &= a_0^2 + 2a_0a_1\kappa + a_1^2\kappa^2. \end{aligned} \tag{18}$$

Substituting Eqs. (18) into Eq. (17), then calculating the coefficient of κ^i to zero, where $i = 1,2,3,4$ yields

$$\kappa^1(\xi):\ -\omega a_1\beta - a_0^2 a_1\beta + \delta a_1\beta^3 = 0, \tag{19}$$

$$\kappa^2(\xi):\ -\omega a_1\eta - a_0^2 a_1\eta - 2a_0 a_1^2\beta + 7\delta a_1\beta^2\eta = 0, \tag{20}$$

$$\kappa^3(\xi):\ -2a_0 a_1^2\eta - a_1^3\beta + 12\delta a_1\beta\eta^2 = 0, \tag{21}$$

$$\kappa^4(\xi):\ -a_1^3\eta + 6\delta a_1\eta^3 = 0 \tag{22}$$

Solving Eq. (19)-(22), we obtain

$$a_0 = \pm\frac{\beta\sqrt{6\delta}}{2},\ a_1 = \pm\sqrt{6\delta}\ \text{ and }\ \omega = -\frac{\beta^2\delta}{2}. \tag{23}$$

The following two precise solutions can be obtained by substituting Eq. (23) into Eq. (17). Then, we utilize the general solutions to the Bernoulli Eqs. (5) and (6). Four exact solutions are got of (16) written in terms of the exponential function.

Case I: if $\beta > 0$ and $\eta < 0$, we get

$$v_1(x,t) = \sqrt{6\delta}\left(\frac{\beta}{2} + \eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}\right)\right),$$

$$v_2(x,t) = -\sqrt{6\delta}\left(\frac{\beta}{2} + \eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}\right)\right),$$

where $\xi = x - \omega t$ and ξ_0 is a constant of the integration.

Case II: if $\beta < 0$ and $\eta > 0$, we get

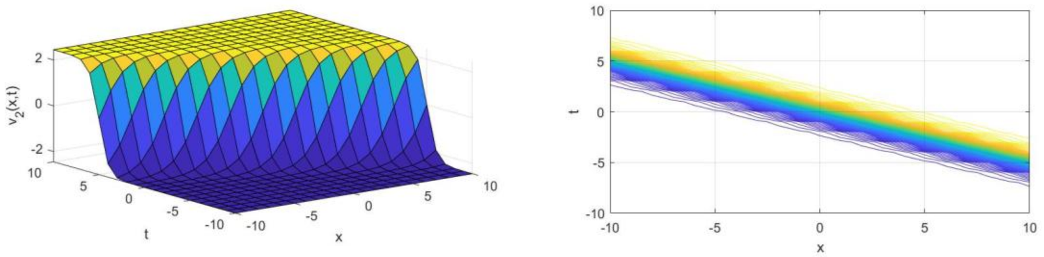
$$v_3(x,t) = \sqrt{6\delta}\left(\frac{\beta}{2} - \eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}\right)\right),$$

$$v_4(x,t) = -\sqrt{6\delta}\left(\frac{\beta}{2} - \eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}\right)\right),$$

where $\xi = x - \omega t$ and ξ_0 is a constant of the integration.

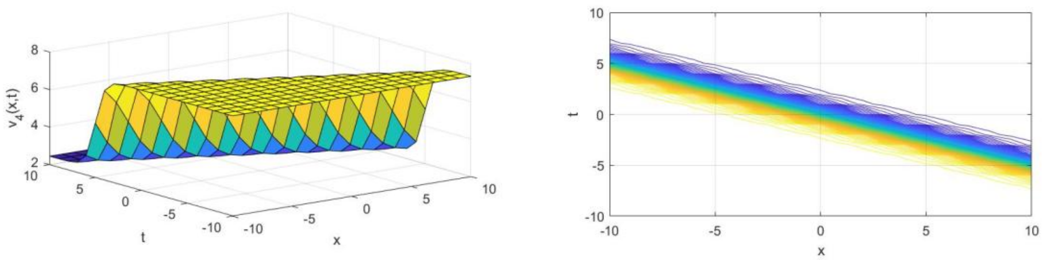
We offer the three-dimensional and contour plots representations of some of the exact solutions, $v_1(x,t)$ in **Case I** and $v_4(x,t)$ in **Case II**, in Figures 3 and 4. We set $\beta = \pm 1,\ \eta = \mp 1,\ \delta = 4,\ \omega = -2$ and

$-10 \le x,t \le 10$, we obtain the plots of the selected exact traveling wave solution, which represents the kink wave solutions.



$$v_2(x,t) = -\sqrt{6\delta}\left(\frac{\beta}{2} + \eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}\right)\right)$$

Figure 3 Kink wave solution for $v_2(x,t)$ with $\beta = 1,\ \eta = -1,\ \delta = 4$ and $-10 \le x,t \le 10$



$$v_4(x,t) = -\sqrt{6\delta}\left(\frac{\beta}{2} - \eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}\right)\right)$$

Figure 4 Kink wave solution for $v_4(x,t)$ with $\beta = -1,\ \eta = 1,\ \delta = 4$ **and** $-10 \le x,t \le 10$

4. CONCLUSION

This article represents the way to use the SE method with the Bernoulli to handle two different kinds of nonlinear evolution problems to find The correct solution for traveling wave for the l Phi-Four equation and the modified Korteweg-De Vries equation. The answers are found in the exponential function. By setting up the parameters appropriately, certain unique solutions can be generated. As illustrated in Figures 1–4, the wave effects of the Phi-Four equation and the modified Korteweg-De Vries equation were kink waves.

The SE method with the Bernoulli is simple to comprehend. Also, this research shows that this suggested methodology is suitable and highly practical for finding accurate solutions to the

Phi-Four equation and the modified Korteweg-De Vries equation. It is a reliable and effective method that yields accurate solutions.

The SE method with the Bernoulli has only 4 steps to finding the answer, and you can find the answer and check the answer. This method has been extensively utilized by researchers, including [2-3], [14-16].

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