CLOSED FORM EXACT SOLUTIONS TO THE COMBINED KDV-MKDV EQUATION AND THE (2+1)-DIMENSIONAL GBS EQUATION VIA THE RICCATI SUB-EQUATION METHOD

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ABSTRACT

The main goal of this study is to find exact traveling wave solutions of the combined kdv-mkdv equation and the (2+1)-dimensional generalized breaking soliton equation using the Riccati sub-equation method. The solutions are shown by hyperbolic and trigonometric functions, which can be transformed into kink waves and periodic waves. Their graphical representations are two-dimensional, three-dimensional graphs, and contour graphs are shown using suitable parameter values. Additionally, the results proved that the method employed in this study is a powerful analytical tool for obtaining exact traveling wave solutions to nonlinear models that are used in many different engineering and scientific disciplines.

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Keywords: Riccati sub-equation method; partial differential equation; combined kdv-mkdv equation; (2+1)-dimensional generalized breaking soliton equation

1. INTRODUCTION

Nonlinear partial differential equations have become a useful tool for describing the natural phenomena of science and engineering such as optical fiber communications, atmospheric pollutant dispersion, solid-state physics, signal processing, mechanical engineering, electric control theory, relativity, chemical reactions, etc. Recently, researchers have found many powerful methods to get exact solutions to nonlinear partial differential equations, such as the simple equation method [1], the modified simple equation method [2], the Kudryashov method [3], the (G′/G)-expansion method [4], the sine-Gordon expansion method [5], the Riccati sub-equation method [6], the Riccati-Bernoulli sub-ODE method [7], the Laplace optimized decomposition method [8], the new extended direct algebraic method [9], etc.

In the present work, we take into consideration the combined kdv-mkdv equation [10],

$$u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx} = 0, \tag{1.1}$$

where $u = u(x,t)$, α and β are real constants. Many analytical techniques have been used to study this equation. From numerous authors, such as in 2010 using the Jacobi elliptic functions expansion method [11], in 2012 using the improved (G′/G)-expansion method [12], in 2014 using the complex method [13], in 2016 using the consistent tanh expansion (CTE) method [14], and in 2023 using the Bernoulli sub-ODE method [10]. And we investigate the (2+1)-dimensional generalized breaking soliton (GBS) equation [15],

$$u_{xt} + 4u_{xx}u_y + 4u_x u_{xy} + u_{xxxy} = 0, \tag{1.2}$$

where $u = u(x,y,t)$ In many papers, the (2+1)-dimensional GBS equation (1.2) was investigated with various techniques, such as in 2010 using the generalized Jacobi elliptic function method [15], in 2013 using the new generalized (G'/G)-expansion method [16], in 2015 using the modified simple equation method [17], in 2018 using the Bell's polynomials method [18].

In this work, we use the traveling wave to transform the combined kdv-mkdv equation and the (2+1)-dimensional generalized breaking soliton (GBS) equation into nonlinear ordinary differential equations. Then, using the Riccati sub-equation method, we have displayed the analytical solutions, and the wave effects are shown in 2D, 3D, and contour graphs.

2. DESCRIPTION OF THE RICCATI SUB-EQUATION METHOD

This section presents the Riccati sub-equation method, a simple technique for finding traveling wave solutions to nonlinear partial differential equations. Assume that the nonlinear partial differential equation with two independent variables x and t is represented by:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt} \ldots) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function and P is a polynomial in $u(x,t)$ and its various partial derivatives, in which highest-order derivatives and nonlinear terms are involved. The following five processes can be used in the Riccati sub-equation method [19,20].

First, transformation process.
Denoting the traveling wave solution of PDE (2.1) as:

$$u(x,t) = u(\xi),\ \xi = x - \omega t, \tag{2.2}$$

where ω is the speed of the traveling wave. We are capable of transforming Eq. (2.1) into an ordinary differential equation (ODE) for $u = u(\xi)$ using the traveling wave transformation Eq. (2.2). This enables us to use the following changes:

$$u_x = u',\ \ u_t = -\omega u',\ \ u_{xt} = -\omega u'',\ \ u_{tt} = \omega^2 u'', \ldots$$

then Eq. (2.1) reduces to a nonlinear ordinary differential equation ODE:

$$G(u, u', u'', \ldots) = 0, \tag{2.3}$$

where G is a polynomial of $u(\xi)$ and its derivatives, where the prime represents the derivative with respect to ξ.

Second, solution-assuming process.
Assume the solution of Equation (2.3) in finite series,

$$u(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi), \tag{2.4}$$

when a_i are constants and a_N is non-zero, while $\varphi(\xi)$ satisfies the following the Riccati equation:

Third, general solutions of the Riccati sub equation method.

The Riccati equation method [19,20] is used to find φ, as shown below:

$$\varphi'(\xi)=\sigma+\varphi^2(\xi), \tag{2.5}$$

Where σ is an arbitrary constant. Here, prime denotes the derivation with respect to ξ. By using the general solutions of Eq. (2.5), we obtain the following expression:

Case I: When $\sigma<0$,

$$\varphi_1(\xi)=-\sqrt{-\sigma}\tanh_{pq}\left(\sqrt{-\sigma}\xi\right), \tag{2.6}$$

$$\varphi_2(\xi)=-\sqrt{-\sigma}\coth_{pq}\left(\sqrt{-\sigma}\xi\right), \tag{2.7}$$

$$\varphi_3(\xi)=-\sqrt{-\sigma}\tanh_{pq}\left(2\sqrt{-\sigma}\xi\right)\pm i\sqrt{-\sigma}\,\mathrm{sech}_{pq}\left(2\sqrt{-\sigma}\xi\right), \tag{2.8}$$

$$\varphi_4(\xi)=-\sqrt{-\sigma}\coth_{pq}\left(2\sqrt{-\sigma}\xi\right)\pm\sqrt{-\sigma}\,\mathrm{csch}_{pq}\left(2\sqrt{-\sigma}\xi\right), \tag{2.9}$$

$$\varphi_5(\xi)=-\frac{1}{2}\left(\sqrt{-\sigma}\tanh_{pq}\left(\frac{\sqrt{-\sigma}}{2}\xi\right)+\sqrt{-\sigma}\coth_{pq}\left(\frac{\sqrt{-\sigma}}{2}\xi\right)\right), \tag{2.10}$$

$$\varphi_6(\xi)=\frac{\sqrt{-\left(R^2+S^2\right)\sigma}-R\sqrt{-\sigma}\cosh_{pq}\left(2\sqrt{-\sigma}\xi\right)}{R\sinh_{pq}\left(2\sqrt{-\sigma}\xi\right)+S}, \tag{2.11}$$

$$\varphi_7(\xi)=-\frac{\sqrt{-\left(S^2-R^2\right)\sigma}-R\sqrt{-\sigma}\sinh_{pq}\left(2\sqrt{-\sigma}\xi\right)}{R\cosh_{pq}\left(2\sqrt{-\sigma}\xi\right)+S}, \tag{2.12}$$

where R,S are two non-zero real constants and satisfy $S^2-R^2>0$.

Case II: When $\sigma>0$,

$$\varphi_8(\xi)=\sqrt{\sigma}\tan_{pq}\left(\sqrt{\sigma}\xi\right), \tag{2.13}$$

$$\varphi_9(\xi)=-\sqrt{\sigma}\cot_{pq}\left(\sqrt{\sigma}\xi\right), \tag{2.14}$$

$$\varphi_{10}(\xi)=-\sqrt{\sigma}\tan_{pq}\left(2\sqrt{\sigma}\xi\right)\pm\sqrt{\sigma}\sec_{pq}\left(2\sqrt{\sigma}\right), \tag{2.15}$$

$$\varphi_{11}(\xi)=-\sqrt{\sigma}\cot_{pq}\left(2\sqrt{\sigma}\xi\right)\pm\sqrt{\sigma}\csc_{pq}\left(2\sqrt{\sigma}\xi\right), \tag{2.16}$$

$$\varphi_{12}(\xi)=\frac{1}{2}\left(\sqrt{\sigma}\tan_{pq}\left(\frac{\sqrt{\sigma}}{2}\xi\right)-\sqrt{\sigma}\cot_{pq}\left(\frac{\sqrt{\sigma}}{2}\xi\right)\right), \tag{2.17}$$

$$\varphi_{13}(\xi)=\frac{\pm\sqrt{\left(R^2-S^2\right)\sigma}-R\sqrt{\sigma}\cos_{pq}\left(2\sqrt{\sigma}\xi\right)}{R\sin_{pq}\left(2\sqrt{\sigma}\xi\right)+S}, \tag{2.18}$$

$$\varphi_{14}(\xi) = -\frac{\pm\sqrt{\left(R^2 - S^2\right)\sigma} - R\sqrt{\sigma}\sin_{pq}\left(2\sqrt{\sigma}\xi\right)}{R\cos_{pq}\left(2\sqrt{\sigma}\xi\right) + S}, \tag{2.19}$$

where R, S are two non-zero real constants and satisfy $R^2 - S^2 > 0$.

Case III: When $\sigma = 0$

$$\varphi_{15}(\xi) = -\frac{1}{\xi + g}, \qquad g = const \tag{2.20}$$

The different types of generalized hyperbolic functions are defined as follows [19,20], with p and q arbitrary constants, $p > 0, q > 0$,

$$\sinh_{pq}(\theta) = \frac{pe^{\theta} - qe^{-\theta}}{2}, \tag{2.21}$$

$$\cosh_{pq}(\theta) = \frac{pe^{\theta} + qe^{-\theta}}{2}, \tag{2.22}$$

$$\tanh_{pq}(\theta) = \frac{pe^{\theta} - qe^{-\theta}}{pe^{\theta} + qe^{-\theta}}, \tag{2.23}$$

$$\coth_{pq}(\theta) = \frac{pe^{\theta} + qe^{-\theta}}{pe^{\theta} - qe^{-\theta}}, \tag{2.24}$$

$$\operatorname{sech}_{pq}(\theta) = \frac{2}{pe^{\theta} + qe^{-\theta}}, \tag{2.25}$$

$$\operatorname{csch}_{pq}(\theta) = \frac{2}{pe^{\theta} - qe^{-\theta}}, \tag{2.26}$$

where θ is an independent variable.

The different types of generalized triangular functions are defined as follows [19,20], with p and q arbitrary constants, $p > 0, q > 0$,

$$\sin_{pq}(\theta) = \frac{pe^{i\theta} - qe^{-i\theta}}{2i}, \tag{2.27}$$

$$\cos_{pq}(\theta) = \frac{pe^{i\theta} + qe^{-i\theta}}{2}, \tag{2.28}$$

$$\tan_{pq}(\theta) = -i\frac{pe^{i\theta} - qe^{-i\theta}}{pe^{i\theta} + qe^{-i\theta}}, \tag{2.29}$$

$$\cot_{pq}(\theta) = i\frac{pe^{i\theta} + qe^{-i\theta}}{pe^{i\theta} - qe^{-i\theta}}, \tag{2.30}$$

$$\sec_{pd}(\theta)=\frac{2}{pe^{i\theta}+qe^{-i\theta}}, \quad (2.31)$$

$$\csc_{pq}(\theta)=\frac{2i}{pe^{i\theta}-qe^{-i\theta}}, \quad (2.32)$$

where θ is an independent variable.

Fourth, the N exploration process.

The positive integer N can be determined by balancing the highest-order derivative and the highest nonlinear terms in Eq. (2.3).

Fifth, the solution-seeking process.

Substituting Eqs. (2.4) and (2.5) into Eq. (2.3), the coefficients of all terms of the same order $\varphi^{i}\,(i=0,1,2,3,...)$ are gathered, and the coefficients are set to zero. We get an overdetermined system of algebraic equations with respect to $a_i\,(i=0,1,2,...,N)$. When all the parameters in Eq. (2.4) are substituted, the solutions to Eq. (2.1) for the traveling wave are reached.

3. DESCRIPTIONS OF THE RICCATI SUB-EQUATION METHOD

Next, we wish to apply the preceding Ricciti sub-equation method to solve both the combined kdv-mkdv equation and the (2+1)-dimensional GBS equation:

3.1 Results of the combined kdv-mkdv equation the combined kdv-mkdv equation

The combined kdv-mkdv equation is

$$u_t+\alpha uu_x+\beta u^2u_x+u_{xxx}=0, \quad (3.1.1)$$

where α and β are real constants. We will reduce it to an ODE using the traveling wave variable $\xi=x-\omega t$. The substitution of the transformation into Eq. (3.1.1) leads to:

$$-\omega u'+\alpha uu'+\beta u^2u'+u'''=0. \quad (3.1.2)$$

Integration Eq. (3.1.2) with the zero constant, we get:

$$-\omega u+\frac{\alpha u^2}{2}+\frac{\beta u^3}{3}+u''=0. \quad (3.1.3)$$

Next, we utilized the balance approach of the highest-order derivative term and the highest nonlinear terms of Eq. (3.1.3). then were N equals 1. We have the solution to Eq. (3.1.3) as follows:

$$u(\xi) = a_0 + a_1\varphi, \tag{3.1.4}$$

where φ satisfies Eq. (2.5). Therefore, the expressions for u'', u^2 and u^3 are expressed as:

$$\begin{aligned} u'' &= 2\sigma a_1\varphi + 2a_1\varphi^3, \\ u^2 &= a_0^2 + 2a_0a_1\varphi + a_1^2\varphi^2, \\ u^3 &= a_0^3 + 3a_0^2a_1\varphi + 3a_0a_1^2\varphi^2 + a_1^3\varphi^3. \end{aligned} \tag{3.1.5}$$

Substituting Eqs. (3.1.4) and (3.1.5) into Eq. (3.1.3) and collecting the coefficients of φ^i where $i = 0, 1, 2, 3,$ yields

$$\begin{aligned} &\varphi^0; \quad -\omega a_0 + \frac{\alpha a_0^2}{2} + \frac{\beta a_0^3}{3} = 0, \\ &\varphi^1; \quad -\omega a_1 + \alpha a_0 a_1 + \beta a_0^2 a_1 + 2\sigma a_1 = 0, \\ &\varphi^2; \quad \frac{\alpha a_1^2}{2} + \beta a_0 a_1^2 = 0, \\ &\varphi^3; \quad \frac{\beta a_1^3}{3} + 2a_1 = 0. \end{aligned} \tag{3.1.6}$$

Solving these equations, we get

$$a_0 = -\frac{\alpha}{2\beta}, \quad a_1 = \pm\sqrt{-\frac{6}{\beta}}, \quad \sigma = \frac{\alpha^2}{24\beta}, \quad \omega = \frac{-\alpha^2}{6\beta} \tag{3.1.7}$$

The following are the exact traveling wave solutions to the combined kdv-mkdv equation:

Case I: when $\sigma < 0$,

$$u_1(x,t) = \frac{\alpha}{2\beta}\left(-1 \pm \tanh_{pq}\left(\sqrt{-\frac{\alpha^2}{24\beta}}\left(x + \frac{\alpha^2}{6\beta}t\right)\right)\right), \tag{3.1.8}$$

$$u_2(x,t) = \frac{\alpha}{2\beta}\left(-1 \pm \coth_{pq}\left(\sqrt{-\frac{\alpha^2}{24\beta}}\left(x + \frac{\alpha^2}{6\beta}t\right)\right)\right), \tag{3.1.9}$$

$$u_3(x,t) = \frac{\alpha}{2\beta}\left(-1 \pm \tanh_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x + \frac{\alpha^2}{6\beta}t\right)\right) \pm i\,\mathrm{sech}_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x + \frac{\alpha^2}{6\beta}t\right)\right)\right), \tag{3.1.10}$$

$$u_4(x,t) = \frac{\alpha}{2\beta}\left(-1 \pm \coth_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x + \frac{\alpha^2}{6\beta}t\right)\right) \pm \mathrm{csch}_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x + \frac{\alpha^2}{6\beta}t\right)\right)\right), \tag{3.1.11}$$

$$u_5(x,t)=\frac{\alpha}{2\beta}\left(-1\pm\frac{1}{2}\left(\tanh_{pq}\left(\frac{1}{2}\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)+\coth_{pq}\left(\frac{1}{2}\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right)\right), \quad (3.1.12)$$

$$u_6(x,t)=\frac{\alpha}{2\beta}\left(-1\pm\left(\frac{\sqrt{R^2+S^2}-R\cosh_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)}{R\sinh_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)+S}\right)\right), \quad (3.1.13)$$

$$u_7(x,t)=\frac{\alpha}{2\beta}\left(-1\mp\left(\frac{\sqrt{S^2-R^2}-R\sinh_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)}{R\cosh_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)+S}\right)\right), \quad (3.1.14)$$

where R,S are two non-zero real constants and satisfy $S^2-R^2>0$.

Case II: When $\sigma>0$,

$$u_8(x,t)=\frac{\alpha}{2\beta}\left(-1\pm i\tan_{pq}\left(\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right), \quad (3.1.15)$$

$$u_9(x,t)=\frac{\alpha}{2\beta}\left(-1\pm i\cot_{pq}\left(\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right), \quad (3.1.16)$$

$$u_{10}(x,t)=\frac{\alpha}{2\beta}\left(-1\pm i\left(\tan_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\pm\sec_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right)\right), \quad (3.1.17)$$

$$u_{11}(x,t)=\frac{\alpha}{2\beta}\left(-1\pm i\left(\cot_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\pm\csc_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right)\right), \quad (3.1.18)$$

$$u_{12}(x,t)=\frac{\alpha}{2\beta}\left(-1\pm\frac{1}{2}i\left(\tan_{pq}\left(\frac{1}{2}\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)-\cot_{pq}\left(\frac{1}{2}\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right)\right), \quad (3.1.19)$$

$$u_{13}(x,t)=\frac{\alpha}{2\beta}\left(-1\pm i\left(\frac{\pm\sqrt{R^2-S^2}-R\cos_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)}{R\sin_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)+S}\right)\right), \quad (3.1.20)$$

$$u_{14}(x,t)=\frac{\alpha}{2\beta}\left(-1\mp i\left(\frac{\pm\sqrt{R^2-S^2}-R\sin_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)}{R\cos_{pq}\left(2\sqrt{\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)+S}\right)\right), \quad (3.1.21)$$

where R, S are two non-zero real constants and satisfy $R^2-S^2>0$.

3.1 Results of the (2+1)-dimensional GBS equation

The (2+1)-dimensional GBS equation is

$$v_{xt}+4v_{xx}v_y+4v_xv_{xy}+v_{xxxy}=0. \quad (3.2.1)$$

We will reduce it to an ODE using the traveling wave variable $\xi=x+y-\omega t$. The substitution of the transformation into Eq. (3.2.1) leads to:

$$-\omega v''+8v'v''+v^{(4)}=0. \quad (3.2.2)$$

Integration Eq. (3.2.2) with the zero constant, we get:

$$-\omega v'+4(v')^2+v'''=0. \quad (3.2.3)$$

Next, we utilized the balance approach of the highest-order derivative term and the highest nonlinear terms of Eq. (3.2.3). then were N equals 1. We have the solution to Eq. (3.2.3) as follows:

$$u(\xi)=a_0+a_1\varphi, \quad (3.2.4)$$

where φ satisfies Eq. (2.5). Therefore, the expressions for $v', (v')^2$ and v''' are expressed as:

$$\begin{aligned} v'&=\sigma a_1+a_1\varphi^2,\\ (v')^2&=\sigma^2a_1^2+2\sigma a_1^2\varphi^2+a_1^2\varphi^4,\\ v'''&=2\sigma^2a_1+8\sigma a_1\varphi^2+6a_1\varphi^4, \end{aligned} \quad (3.2.5)$$

Substituting Eq. (3.2.5) into Eq. (3.2.2) and collecting the coefficients of φ^{i} where $i=0,2,4$ yields

$$\begin{aligned}\varphi^{0};&\quad 2\sigma^{2}a_{1}+4\sigma^{2}a_{1}^{2}-\omega\sigma a_{1}=0,\\ \varphi^{2};&\quad 8\sigma a_{1}+8\sigma a_{1}^{2}-\omega a_{1}=0,\\ \varphi^{4};&\quad 6a_{1}+4a_{1}^{2}=0.\end{aligned} \tag{3.2.6}$$

Solving these equations, we get

$$a_{1}=-\frac{3}{2}\quad\text{and}\quad\sigma=\frac{-\omega}{4} \tag{3.2.7}$$

The following are the exact traveling wave solutions to the

Case I: when $\sigma<0$,

$$v_{1}(x,y,t)=a_{0}+\frac{3}{4}\sqrt{\omega}\tanh_{pq}\left(\frac{\sqrt{\omega}}{2}(x+y-\omega t)\right), \tag{3.2.8}$$

$$v_{2}(x,y,t)=a_{0}+\frac{3}{4}\sqrt{\omega}\coth_{pq}\left(\frac{\sqrt{\omega}}{2}(x+y-\omega t)\right), \tag{3.2.9}$$

$$v_{3}(x,y,t)=a_{0}+\frac{3\sqrt{\omega}}{4}\left(\tanh_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)\pm i\,\mathrm{sech}_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)\right), \tag{3.2.10}$$

$$v_{4}(x,y,t)=a_{0}+\frac{3\sqrt{\omega}}{4}\left(\coth_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)\pm \mathrm{csch}_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)\right), \tag{3.2.11}$$

$$v_{5}(x,y,t)=a_{0}+\frac{3\sqrt{\omega}}{8}\left(\tanh_{pq}\left(\frac{\sqrt{\omega}}{4}(x+y-\omega t)\right)+\coth_{pq}\left(\frac{\sqrt{\omega}}{4}(x+y-\omega t)\right)\right), \tag{3.2.12}$$

$$v_{6}(x,y,t)=a_{0}-\frac{3\sqrt{\omega}}{4}\left(\frac{\sqrt{R^{2}+S^{2}}-R\cosh_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)}{R\sinh_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)+S}\right), \tag{3.2.13}$$

$$v_{7}(x,y,t)=a_{0}+\frac{3\sqrt{\omega}}{4}\left(\frac{\sqrt{S^{2}-R^{2}}+R\sinh_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)}{R\cosh_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)+S}\right), \tag{3.2.14}$$

where R,S are two non-zero real constants and satisfy $S^{2}-R^{2}>0$.

Case II: When $\sigma>0$,

$$v_{8}(x,y,t)=a_{0}-\frac{3}{4}\sqrt{-\omega}\tan_{pq}\left(\frac{\sqrt{-\omega}}{2}(x+y-\omega t)\right), \tag{3.2.15}$$

$$v_9(x,y,t)=a_0+\frac{3}{4}\sqrt{-\omega}\cot_{pq}\left(\frac{\sqrt{-\omega}}{2}(x+y-\omega t)\right), \tag{3.2.16}$$

$$v_{10}(x,y,t)=a_0-\frac{3\sqrt{-\omega}}{4}\left(\tan_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)\pm\sec_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)\right), \tag{3.2.17}$$

$$v_{11}(x,y,t)=a_0+\frac{3\sqrt{-\omega}}{4}\left(\cot_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)\pm\csc_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)\right), \tag{3.2.18}$$

$$v_{12}(x,y,t)=a_0-\frac{3\sqrt{-\omega}}{8}\left(\tan_{pq}\left(\frac{\sqrt{-\omega}}{4}(x+y-\omega t)\right)-\cot_{pq}\left(\frac{\sqrt{-\omega}}{4}(x+y-\omega t)\right)\right), \tag{3.2.19}$$

$$v_{13}(x,y,t)=a_0-\frac{3\sqrt{-\omega}}{4}\left(\frac{\pm\sqrt{R^2-S^2}-R\cos_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)}{R\sin_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)+S}\right), \tag{3.2.20}$$

$$v_{14}(x,y,t)=a_0+\frac{3\sqrt{-\omega}}{4}\left(\frac{\pm\sqrt{R^2-S^2}-R\sin_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)}{R\cos_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)+B}\right), \tag{3.2.21}$$

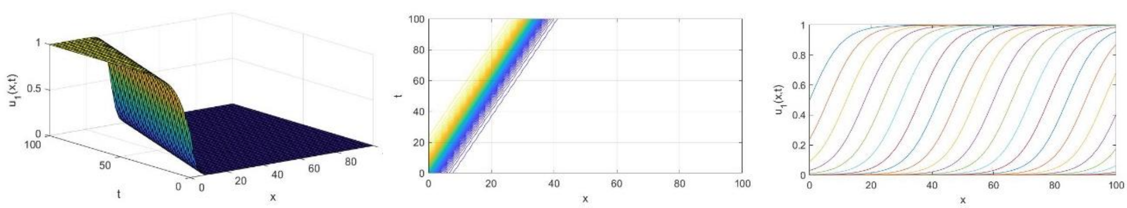
where R, S are two non-zero real constants and satisfy $R^2-S^2>0$.

4. GRAPHICAL REPRESENTATION OF SOME OBTAINED SOLUTION

In this section, we have presented some physical graphs of some solutions to the combined kdv-mkdv equation and the GBS equation.

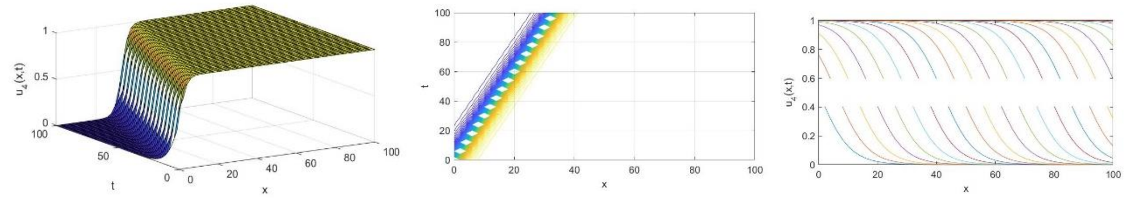
4.1 Graphical representation of the combined kdv- mkdv equation

We set some parameters to get the example graph of the wave effects of the combined kdv-mkdv equation by $\alpha=2, \beta=-2$ in the interval $0\leq x,t\leq 100$, which displayed in Figures 1 and 2, it produces a kink wave solution.



$$u_1(x,t)=\frac{\alpha}{2\beta}\left(-1+\tanh_{pq}\left(\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right),$$

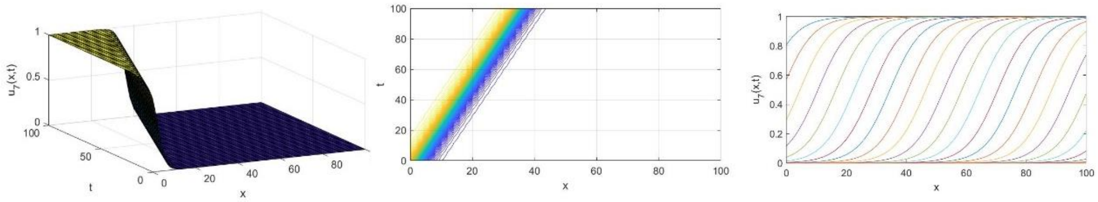
Figure 1. The kink wave solution of $u_1(x,t)$ in 3D, 2D and contour



$$u_4(x,t)=\frac{\alpha}{2\beta}\left(-1\pm\coth_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)-\operatorname{csch}_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)\right),$$

Figure 2. The kink wave solution of $u_4(x,t)$ in 3D, 2D and contour

The graphs of $u_7(x,t)$ by $\alpha=2, \beta=-2, R=1, S=2$ in the interval $0\le x,t\le 100,$ as shown in Figures 3, are the shapes of kink waves that rise or descend from one asymptotic state to another.

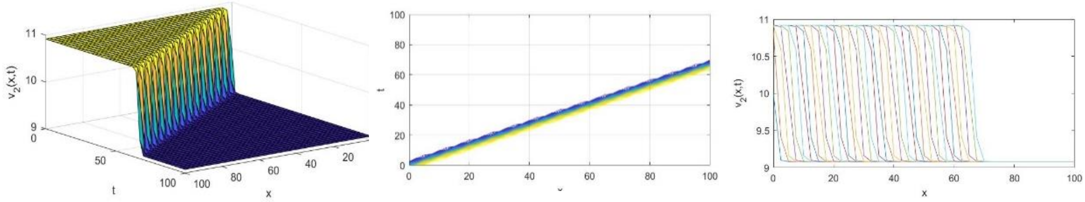


$$u_7(x,t)=\frac{\alpha}{2\beta}\left(-1-\left(\frac{\sqrt{S^2-R^2}-R\sinh_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)}{R\cosh_{pq}\left(2\sqrt{-\frac{\alpha^2}{24\beta}}\left(x+\frac{\alpha^2}{6\beta}t\right)\right)+S}\right)\right),$$

Figure 3. The kink wave solution of $u_7(x,t)$ in 3D, 2D and contour

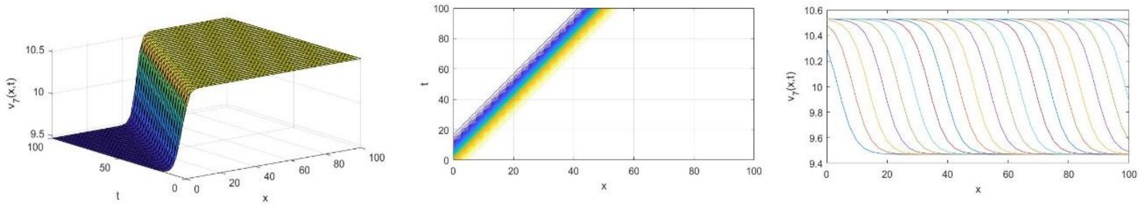
4.2 Graphical representation of the (2+1)-dimensional GBS equation

Next, we represent the shape of the solution to the (2+1)-dimensional GBS equation by setting some parameters $a_0=10$ and $\omega=2.5$ in the interval $0\le x,t\le 100$ for $y=0,$ as displayed in Figure 4. The graphs of $v_7(x,y,t)$ by setting $a_0=10, \omega=0.5, R=2$ and $S=4$ in the interval $0\le x,t\le 100$ for $y=0,$ shown in Figure 5, produce a kink wave solution.



$$v_2(x,y,t)=a_0+\frac{3}{4}\sqrt{\omega}\coth_{pq}\left(\frac{\sqrt{\omega}}{2}(x+y-\omega t)\right),$$

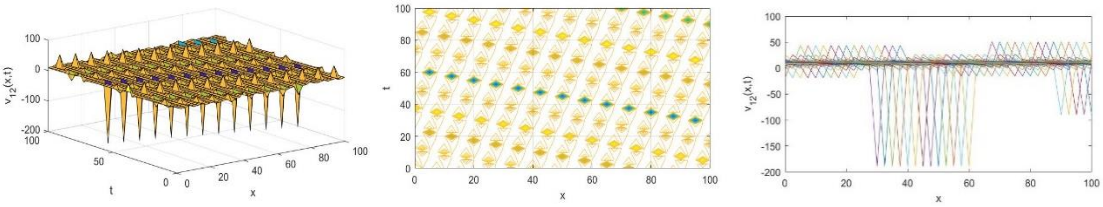
Figure 4. The kink wave solution of $v_2(x,y,t)$ in 3D, 2D and contour



$$v_7(x,y,t)=a_0+\frac{3\sqrt{\omega}}{4}\left(\frac{\sqrt{S^2-R^2}+R\sinh_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)}{R\cosh_{pq}\left(\sqrt{\omega}(x+y-\omega t)\right)+S}\right),$$

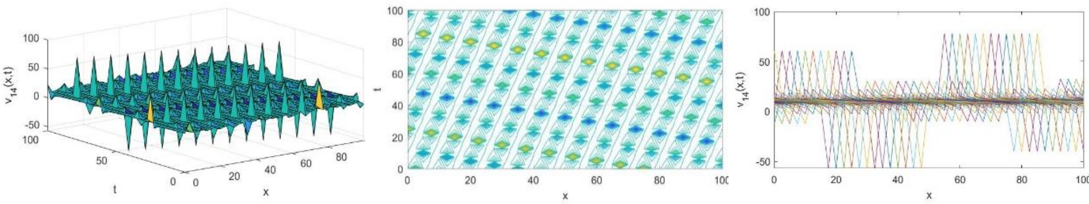
Figure 5. The kink wave solution of $v_7(x,y,t)$ in 3D, 2D and contour

Solution $v_{12}(x,y,t)$ with $a_0=10$ and $\omega=-3$ in the interval $0\le x,t\le 100$ for $y=0$ corresponds to Figure 6, and solution $v_{14}(x,y,t)$ with $a_0=10,\omega=-3,R=4$ and $S=2$ in the interval $0\le x,t\le 100$ for $y=0$ corresponds to Figure 7. All of them produce a periodic traveling wave solution.



$$v_{12}(x,y,t)=a_0-\frac{3\sqrt{-\omega}}{8}\left(\tan_{pq}\left(\frac{\sqrt{-\omega}}{4}(x+y-\omega t)\right)-\cot_{pq}\left(\frac{\sqrt{-\omega}}{4}(x+y-\omega t)\right)\right),$$

Figure 6. The kink wave solution of $v_{12}(x,y,t)$ in 3D, 2D and contour



$$v_{14}(x,y,t)=a_0+\frac{3\sqrt{-\omega}}{4}\left(\frac{\pm\sqrt{R^2-S^2}-R\sin_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)}{R\cos_{pq}\left(\sqrt{-\omega}(x+y-\omega t)\right)+S}\right),$$

Figure 7. The kink wave solution of $v_{14}(x,y,t)$ in 3D, 2D and contour

5. CONCLUSION

In this work, we have examined the combined kdv-mkdv equation and the (2+1)-dimensional generalized breaking soliton equation by means of the efficient technique known as the Riccati sub-equation method. The solutions are found in trigonometric and hyperbolic forms.

The Riccati sub-equation method is powerful and gives the exact traveling wave solutions to the combined kdv-mkdv equation and the (2+1)-dimensional generalized breaking soliton equation. And the Riccati sub-equation method can be used for many other nonlinear partial differential equations to get feasible solutions to tangible incidents.

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