A connection between hybrid one-sided hyperideals and hybrid bi-hyperideals in hypersemigroups

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Abstract

The concept of hybrid structures, which combine soft and fuzzy sets, offers a powerful mathematical framework. In this study, we apply hybrid structures to explore the relationship of hybrid hyperideals in hypersemigroups, a hyperalgebraic structure. Specifically, we focus on understanding and establishing connections between hybrid left (right) hyperideals and hybrid bi-hyperideals in hypersemigroups.

Keywords: hybrid structures; hypersemigroups; hybrid left (right) hyperideals; hybrid bi-hyperideals.

1. Introduction

In 1934, Marty [1] introduced the concept of hyperoperations, which extended the traditional notion of operations by defining the multiplication of two elements as a set rather than a single element. This groundbreaking introduction laid the foundation for developing hyperalgebras, as hyperoperations have extended diverse algebraic structures into hyperalgebras. Consequently, hyperalgebraic systems have found utility in studying various scientific disciplines, including biology, chemistry, and computer science [2-5].

Hypersemigroups, also known as semihypergroups or multisemigroups in certain literature, are a useful concept of hyperstructures [6-8]. They are a generalization of semigroups. Hypersemigroups extend semigroups' fundamental properties, providing a broader framework for mathematical analysis and applications. Hyperideals play a pivotal role in investigating hypersemigroups, particularly in the classification of hypersemigroups using linear inequalities. Hasankhani [9] extensively studied the fundamental properties of left and right hyperideals in hypersemigroups and explored Green's relations of hypersemigroups. These investigations highlight the significance of hyperideals in semihypergroups. In 2015, Changphas and Davvaz [10] introduced the notion of bi-hyperideals in ordered hypersemigroups, which can be viewed as a generalization of hypersemigroups. However, it is worth noting that bi-hyperideals can be defined and studied naturally within the framework of hypersemigroups. Bi-hyperideals represent an extension of left and right hyperideals in hypersemigroups. While this study does not cover all the types of hyperideals, the readers can find more information on the subject in the references [11-14] regarding hyperideals in hypersemigroups.

Hybrid structures combine soft and fuzzy sets and serve as mappings that can effectively handle detailed information. Jun et al. [15] initially introduced the concept of hybrid structures in 2018 and applied them to the study of logical algebras such as BCK- and BCI-algebras. Subsequently, hybrid structures found applications in ordered semigroups [16]. In 2022, the concept of hybrid structures was extended to the study of hypersemigroups. Lekkoksung et al. [17] defined hybrid left (right) hyperideals in hypersemigroups and established some fundamental properties. Furthermore, Mekwian et al. [18] introduced hybrid bi-hyperideals in hypersemigroups and explored their basic properties. The investigation of hybrid structures in ordered hypersemigroups was also started by Sanpan et al. [19] in 2023.

Previous studies have explored the concepts of hybrid left (right) hyperideals and hybrid bi-hyperideals in hypersemigroups. However, no connection has been established between these hybrid hyperideals. Consequently, the primary objective of this article is to bridge this gap and provide a comprehensive understanding of the relationship between hybrid left (right) hyperideals and hybrid bi-hyperideals in hypersemigroups.

2. Preliminaries

A *hyperoperation* $\circ$ on a nonempty set H is a mapping $\circ: H\times H\rightarrow \mathcal{P}^{*}(H)$, where $\mathcal{P}^{*}(H)$ is the set of all subsets of H without the empty set. A hyperoperation $\circ$ on H induces an operation $\hat{\circ}: \mathcal{P}(H)\times\mathcal{P}(H)\rightarrow\mathcal{P}(H)$ defined by

$$A\hat{\circ}B=\begin{cases}\bigcup\limits_{a\in A,b\in B}(a\circ b) & \text{if } A,B\neq\varnothing\\ \varnothing & \text{otherwise}\end{cases}$$

for all $A,B\subseteq H$.

A structure $\langle H;\circ\rangle$ comprising of a nonempty set H and a hyperoperation on H is called a *hypersemigroup* [7] if $A\hat{\circ}(B\hat{\circ}C)=(A\hat{\circ}B)\hat{\circ}C$ for all $A,B,C\subseteq H$.

The following observation is not difficult to verify.

Theorem 2.1 [18]. *If* $\langle H;\circ\rangle$ *is a hypersemigroup, then* $\langle\mathcal{P}(H);\hat{\circ}\rangle$ *is a semigroup.*

From now on, we denote a hypersemigroup $\langle H;\circ\rangle$ by its boldface letter $\mathbf{H}$ of the underlying set, and write the product $A\hat{\circ}B$ by AB for all $A,B\subseteq H$. In particular, if $A=\{a\}$, then we denote AB by aB Similarly, if $B=\{b\}$, then we denote AB by Ab. Moreover, we denote the n-product $A\hat{\circ}\cdots\hat{\circ}A$ of A by A^{n}.

Example 2.2 [20]. *Let* $H=\{a,b,c,d,e\}$. *Define a hyperoperation* $\circ$ *on* H *as follows.*

$\circ$	a	b	c	d	e
a	a	a	$\{a,b,c\}$	a	$\{a,b,c\}$
b	a	a	$\{a,b,c\}$	a	$\{a,b,c\}$
c	a	a	$\{a,b,c\}$	a	$\{a,b,c\}$
d	$\{a,b,d\}$	$\{a,b,d\}$	H	$\{a,b,d\}$	H
e	$\{a,b,d\}$	$\{a,b,d\}$	H	$\{a,b,d\}$	H

By careful calculation, we obtain that $\mathbf{H}:=\langle H;\circ\rangle$ *is a hypersemigroup.*

Now, we recall the concept of hybrid structures. In what follows, let I be the closed unit interval and E be a set of parameters.

Let U be a nonempty set. A mapping $f:E\rightarrow \mathcal{P}(U)\times I$ is said to be a *hybrid structure* in E over U (see [9]). We observe that a hybrid structure f can be determined by two mappings: $f_1:E\rightarrow \mathcal{P}(U)$ and $f_2:E\rightarrow I$. That is, f can be considered as a combination of soft set f_1 in E over U and a fuzzy set f_2 in E. Therefore, any hybrid structure f in E over U can be written as $f=(f_1,f_2)$.

We reintroduce the notions of hybrid left (right) hyperideals and hybrid bi-hyperideals in hypersemigroups as follows. Let $\mathbf{H}$ be a hypersemigroup and f a hybrid structure in H over U. Then, f is said to be:

(1) a *hybrid subhypersemigroup* in $\mathbf{H}$ over U if $f_1(x)\cap f_1(y)\subseteq\bigcap_{u\in xy}\{f_1(u)\}$ and $f_2(x)\vee f_2(y)\geq\bigvee_{u\in xy}\{f_2(u)\}$ for all $x,y\in H$;

(2) a *hybrid left hyperideal* in $\mathbf{H}$ over U if $f_1(y)\subseteq\bigcap_{u\in xy}\{f_1(u)\}$ and $f_2(y)\geq\bigvee_{u\in xy}\{f_2(u)\}$ for all $x,y\in H$;

(3) a *hybrid right hyperideal* in $\mathbf{H}$ over U if $f_1(x)\subseteq\bigcap_{u\in xy}\{f_1(u)\}$ and $f_2(x)\geq\bigvee_{u\in xy}\{f_2(u)\}$ for all $x,y\in H$;

(4) a *hybrid bi-hyperideal* in $\mathbf{H}$ over U if f is a hybrid subhypersemigroup such that $f_1(x)\cap f_1(z)\subseteq\bigcap_{u\in xyz}\{f_1(u)\}$ and $f_2(x)\vee f_2(z)\geq\bigvee_{u\in xyz}\{f_2(u)\}$ for all $x,y,z\in H$.

Example 2.3 *Using the multiplication table in Example 2.2, we define a hybrid structure* f *in* H *over* $\mathbb{N}$ *by*

x	$f_1(x)$	$f_2(x)$
a,b,d	$2\mathbb{N}$	0.5
c,e	$4\mathbb{N}$	0.9

for all $x\in H$. *By careful calculation, we obtain that* f *is a hybrid left hyperideal in* $\mathbf{H}$ *over* $\mathbb{N}$. *However,* f *is not a hybrid right hyperideal in* $\mathbf{H}$ *over* $\mathbb{N}$ *since*

$$f_1(a)\not\subseteq\bigcap_{u\in ac}\{f_1(u)\}=f_1(a)\cap f_1(b)\cap f_1(c)$$

and $f_2(a)\not\geq f_2(a)\cap f_2(b)\cap f_2(c)$. *Define a hybrid structure* g *in* H *over* $\mathbb{N}$ *by*

x	$g_1(x)$	$g_2(x)$
a,b,c	$2\mathbb{N}$	0.5
d,e	$4\mathbb{N}$	0.9

for all $x\in H$. *By careful calculation, we obtain that* g *is a hybrid right hyperideal in* $\mathbf{H}$ *over* $\mathbb{N}$. *Define a hybrid structure* h *in* H *over* $\mathbb{N}$ *by*

x	$h_1(x)$	$h_2(x)$
a	$\mathbb{N}$	0.1
b,c	$3\mathbb{N}$	0.4
d,e	$12\mathbb{N}$	0.4

for all $x \in H$. By careful calculation, we obtain that h is a hybrid bi-hyperideal in $\mathbf{H}$ over $\mathbb{N}$. However, h is not a hybrid left hyperideal in $\mathbf{H}$ over $\mathbb{N}$ since

$$h_1(c) \not\supseteq \bigcap_{u \in dc}\{h_1(u)\} = \bigcap_{x \in H}\{h_1(x)\}.$$

3. Result

Example 2.3 highlights an important observation: a hybrid left hyperideal and a hybrid right hyperideal do not necessarily have any relationship. Additionally, a hybrid bi-hyperideal may not be either a hybrid left hyperideal or a hybrid right hyperideal. These distinctions emphasize the distinct nature and independence of these hybrid hyperideals within the context of hypersemigroups. A question arises: Is a hybrid left (right) hyperideal a hybrid bi-hyperideal in hypersemigroups? The following theorem can be addressed in this inquiry.

Theorem 3.1 *Let $\mathbf{H}$ be a hypersemigroup, and f a hybrid structure in H over U. Then, if f is a hybrid left (right) hyperideal in $\mathbf{H}$ over U, then f is a hybrid bi-hyperideal in $\mathbf{H}$ over U.*

Proof. Assume that f is a hybrid left hyperideal in $\mathbf{H}$ over U. Firstly, we show that f is a hybrid subhypersemigroup in $\mathbf{H}$ over U. Let $x, y \in H$. Since f is a hybrid left hyperideal in $\mathbf{H}$ over U,

$$\bigcap_{u \in xy}\{f_1(u)\} \supseteq f_1(y) \supseteq f_1(x) \cap f_1(y)$$ and

$$\bigvee_{u \in xy}\{f_2(u)\} \leq f_2(y) \leq f_2(x) \vee f_2(y).$$

This means that f is a hybrid subhypersemigroup in $\mathbf{H}$ over U. Next, we let $x, y, x \in H$. Consider

$$\bigcap_{u \in xyz}\{f_1(u)\} = \bigcap_{u \in (xy)z}\{f_1(u)\} \supseteq f_1(z) \supseteq f_1(x) \cap f_1(z)$$

and

$$\bigvee_{u \in xyz}\{f_2(u)\} = \bigvee_{u \in (xy)z}\{f_2(u)\} \leq f_2(z) \leq f_2(x) \vee f_2(y).$$

Therefore, f is a hybrid bi-hyperideal in $\mathbf{H}$ over U. We can illustrate similarly for the case that f is a hybrid right hyperideal in $\mathbf{H}$ over U.

Let $\mathrm{HL}(\mathbf{H}), \mathrm{HR}(\mathbf{H})$ and $\mathrm{HB}(\mathbf{H})$ be the set of all hybrid left hyperideals, hybrid right hyperideals, and hybrid bi-hyperideals in $\mathbf{H}$ over U, respectively. From the analysis presented in Example 2.3 and Theorem 3.1, we can summarize the following key observations:

Corollary 3.2 *Let $\mathbf{H}$ be a hypersemigroup. Then,*

$$\mathrm{HL}(\mathbf{H}) \subseteq \mathrm{HB}(\mathbf{H}) \text{ and } \mathrm{HR}(\mathbf{H}) \subseteq \mathrm{HB}(\mathbf{H}).$$

4. Conclusion

This research focuses on exploring the interrelationships among hybrid hyperideals in hypersemigroups. Our result indicates that every hybrid left (right) hyperideal is a hybrid bi-hyperideal. However, it is important to note that the converse may not always hold. Building upon these results, future investigations will employ these hybrid hyperideals to facilitate the classification of hypersemigroups.

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6. References

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