On fuzzy α-ideals in ordered semigroups

Nuchanat Tiprachot[1], Somsak Lekkoksung[2] and Nareupanat Lekkoksung[2*]

[1] Demonstration School of Khon Kaen University, Faculty of Education, Khon Kaen University, Khon Kaen 40002, Thailand

[2] Division of Mathematics, Faculty of Engineering, Rajamangala University of Technology Isan, Khon Kaen Campus, Khon Kaen 40000, Thailand

*Corresponding Author: nareupanat.le@rmuti.ac.th

Received 14 August 2025; Received in revised form 4 October 2025; Accepted 10 October 2025

Abstract

The concept of fuzzy α-ideals in ordered semigroups is introduced as a generalized concept of several fuzzy ideals. Based on the reduction relations of full words, we provide the interconnection of among various α-ideals in ordered semigroups. Moreover, a description of fuzzy α-ideals in ordered semigroups is provided through an associative binary operation defined on the set of all fuzzy sets.

Keywords: ordered semigroup; fuzzy ideal; fuzzy α-ideal

1. Introduction

The concept of ordered semigroups forms an associative structure that has been extensively studied in recent years. A fundamental tool used in the investigation of ordered semigroups is the concept of *ideals*, of which several types have been introduced to explore structural properties.

One significant generalization of ideals in ordered semigroups is the concept of α-ideals, first introduced by Towwun and Changphas in 2013. They provided a comprehensive description of the fundamental properties of α-ideals in this context (see [1]). Subsequently, in 2022, Tiprachot *et al.* characterized various regularity conditions in ordered semigroups using the framework of α-ideals. In the same study, the interconnections among α-ideals were also examined (see [2]). In 2025, Lekkoksung generalized α-ideals to the notion of partition ideal elements. This notion can illustrate the complexity of the connections of several α-ideals (see [3]).

The concept of *fuzzy sets* was originally introduced by Zadeh in 1965 as a generalization of classical (crisp) sets. While membership in crisp sets is binary (0 or 1), fuzzy sets allow degrees of membership ranging between 0 and 1. This framework has been widely applied in mathematics to generalize various algebraic structures. In particular, fuzzy ideals in ordered semigroups have been studied extensively (see [4]).

In this paper, we introduce the concept of fuzzy α-ideals in ordered semigroups, as a unification and generalization of several existing types of fuzzy ideals. We also investigate the interrelationships among fuzzy α-ideals using the reduction relation on full words as a central tool. We also characterize fuzzy α-ideals in ordered semigroups through a particular associative binary operation.

2. Preliminaries

In this section, we review the notions of full words, including the reduction relations of full words, and ordered semigroups. Some fundamental properties of these concepts are also provided. For more details, the readers can find all the terminologies needed in this paper in [1-3, 5-10]. Throughout this paper, we denote the set of all natural numbers by ℕ, that is, $\mathbb{N} := \{1,2,3,\dots\}$. Moreover, for any $n \in \mathbb{N}$, the set $\{1,2,3,\dots,n\}$ is denoted by $[n]$.

For any set X, we let X^* the *free monoid* over X. The neutral element of X^* is called the *empty word* and is denoted by ε. The *free semigroup* over X is denoted by X^+, that is, $X^+ = X^* \setminus \{\varepsilon\}$. The length of any $\alpha \in X^*$ is denoted by $|\alpha|$, where $|\alpha| = 0$ if $\alpha = \varepsilon$. The free semigroup over $\{0,1\}$ is denoted by B. We define $\mathrm{F} := \{0,1\}^* \setminus (\{0\}^* \cup \{1\}^*)$ to be the set of all *full words*.

Let $C = \{0,1\} \cup \{\varepsilon\}$. We define a totally order $\leq_C$ to be the set containing all identity relations on C including $(\varepsilon,0),(\varepsilon,1)$ and $(0,1)$. Then, we obtain a totally ordered set $\mathbf{C} := \langle C; \leq_C \rangle$.

For any $\alpha \in B$, and $n \in \mathbb{N}$ with $n \geq |\alpha|$, an n-tuple $(\alpha_1, \dots, \alpha_n)$ over C is called a *canonical tuple* of α with length n if $\alpha = \alpha_1 \cdots \alpha_n$, $\alpha_1 \neq \varepsilon$, and, if $\alpha_i \neq \varepsilon$, then $\alpha_{i-1} \neq 1$ for all $1 < i \leq n$. The set of all canonical tuple of α with length n is denoted by $\mathrm{C}_n(\alpha)$.

Let $\alpha, \beta \in \mathrm{B}$ and $n \geq \max(|\alpha|, |\beta|)$. Suppose that $\boldsymbol{u} = (u_1, \dots, u_n)$ and $\boldsymbol{v} = (v_1, \dots, v_n)$ be a canonical tuple of α and β with length n, respectively. We say that $\boldsymbol{u}$ is a *reduction* of $\boldsymbol{v}$, denoted by $\boldsymbol{u} \leq \boldsymbol{v}$ if $u_i \leq_C v_i$ for all $i \in [n]$. In particular, if $\alpha, \beta \in \mathrm{F}$, we say that α is a *reduction* of β, denoted by $\alpha \leq \beta$, if $\boldsymbol{u} \leq \boldsymbol{v}$ for some $\boldsymbol{u} \in \mathrm{C}_{|\beta|}(\alpha)$ and $\boldsymbol{v} \in \mathrm{C}_{|\beta|}(\beta)$ (see [3]).

It was proved that the set of all full words F together with the reduction relation $\leq$ forms a partially ordered set.

Let $\alpha \in \mathrm{B}$. For any $n \geq |\alpha|$ and $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathrm{C}_n(\alpha)$, we denote

$$1_n(\boldsymbol{u}) := \{i \in [n] : u_i = 1\}.$$

When $n = |\alpha|$ and $\boldsymbol{u} \in \mathrm{C}_{|\alpha|}(\alpha)$, we simply write $1(\alpha)$ for $1_n(\boldsymbol{u})$.

An *ordered semigroup* is a mathematical structure $\langle S; \cdot, \leq \rangle$ consisting of a nonempty set S, an associative operation $\cdot$, and a partial order $\leq$ on S such that $\leq$ preserves the operation $\cdot$. By the associativity of the operation $\cdot$, we can write any product of elements S without using parentheses. For any elements $a_1, \dots, a_n \in S$, we simply write the product $a_1 \cdots a_n$ by a_1^n. We usually denote an ordered semigroup $\langle S; \cdot, \leq \rangle$ with the boldface $\mathbf{S}$ of its underlying set.

The binary operation $\cdot$ on an ordered semigroup $\mathbf{S}$ can be extended as follows: for any subsets A and B of S, we define $AB = \{ab : a \in A, b \in B\}$. We note here that $AB = \emptyset$ if A or B is an empty set. Moreover, the partial order of $\mathbf{S}$ induces the operator $(\cdot]$ defined by $(A] = \{x \in S : x \leq a$ for some $a \in A\}$ for every subset A of S. We also note that $(\emptyset] = \emptyset$.

Next, we present some notation used to compute the multiplication of our objects of interest.

Let $\mathbf{S}$ be an ordered semigroup, and $\alpha = \alpha_1 \cdots \alpha_n \in \mathrm{B}$, where $|\alpha| = n$ for some $n \in \mathbb{N}$. A mapping $\overline{\alpha} : S \to S$ is defined by $\overline{\alpha}(A) := X_1 \cdots X_n$ for any $A \subseteq S$, where

$$X_i = \begin{cases} A \text{ if } \alpha_i = 1, \\ S \text{ if } \alpha_i = 0, \end{cases}$$

for all $i \in [n]$.

The concept of α-ideals in ordered semigroups is defined as follows. Let $\mathbf{S}$ be an ordered semigroup, and A a nonempty subset of S such that $(A] \subseteq A$. Then, A is said to be an *α-ideal* of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$, that is, $AA \subseteq A$, and $\overline{\alpha}(A) \subseteq A$. By this definition, we can see that several ideals in ordered semigroups can be viewed as a special case of α-ideals. For example, an (m, n)-ideal is a $1^m 0 1^n$-ideal, and an n-interior ideal is a 01^n0-ideal.

Next, we recall the concept of fuzzy sets and fuzzy ideals in ordered semigroups. Let X be a nonempty set. A mapping $f : X \to [0,1]$ is called a *fuzzy set* of X. We denote the set of all fuzzy sets of X by $\mathrm{F}(X)$. A well-known fuzzy set is the *characteristic function* of A in X, and is defined by, for any $A \subseteq X$,

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ otherwise}, \end{cases}$$

for all $x \in X$. For any $u \in [0,1]$, can be also defined a fuzzy set of X by $u(x) = u$ for all $x \in X$. We define a relation $\subseteq$ on the set $\mathrm{F}(X)$ as follows: $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in X$.

The concept of fuzzy sets can be applied to ordered semigroups as follows. Let $\mathbf{S}$ be an ordered semigroup. For $f \in \mathrm{F}(\mathbf{S})$, we mean a fuzzy set f of the underlying set of $\mathbf{S}$. For any $x \in S$, we define $\mathbf{S}_a := \{(u,v) \in S \times S : a \leq uv\}$. We define a binary operation $\circ$ on $\mathrm{F}(\mathbf{S})$ as follows. For any $f, g \in F(S)$,

$$(f \circ g)(x) = \begin{cases} \bigvee_{(u,v)\in S_x} [f(u) \wedge g(v)] & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in S$. We note here that the operations $\vee$ and $\wedge$ are a supremum and an infimum, respectively.

It was proved in [5, 6] that the operation $\circ$ defined on $\mathrm{F}(\mathbf{S})$ is associative. Therefore, the mapping $\overline{\alpha}$ defined on S can be extended to $\mathrm{F}(\mathbf{S})$ as follows. Let $\alpha = \alpha_1 \cdots \alpha_n \in \mathrm{B}$, where $|\alpha| = n$ for some $n \in \mathbb{N}$. A mapping $\overline{\alpha} : \mathrm{F}(\mathbf{S}) \to \mathrm{F}(\mathbf{S})$ is defined by $\overline{\alpha}(f) := f_1 \circ \cdots \circ f_n$ for any $f \in \mathrm{F}(\mathbf{S})$, where

$$f_i = \begin{cases} f & \text{if } \alpha_i = 1, \\ 1 & \text{if } \alpha_i = 0, \end{cases}$$

for all $i \in [n]$. We note that 1 is a fuzzy set of S assigning any element in S to 1.

Remark 2.1. Let $\mathbf{S}$ be an ordered semigroup, $x \in S$, and $\alpha = \alpha_1 \cdots \alpha_n \in \mathrm{F}$, where $|\alpha| = n$ for some $n \in \mathbb{N}$. We suppose more that that $f \in \mathrm{F}(\mathbf{S})$ and $\mathbf{S}_x \neq \emptyset$. Then, there exist elements $u_1, \dots, u_n = u'_{n-1}, u'_1, \dots, u'_{n-2} \in S$ such that

$$\begin{aligned} &\overline{\alpha}(f)(x) \\ &= \bigvee_{(u_1,u'_1)\in \mathbf{S}_x} [\overline{\alpha_1}(f)(u_1) \wedge \overline{\alpha_2 \cdots \alpha_n}(f)(u'_1)] \\ &= \bigvee_{(u_1,u'_1)\in \mathbf{S}_x} \left[\overline{\alpha_1}(f)(u_1) \wedge \bigvee_{(u_2,u'_2)\in \mathbf{S}_{u'_1}} (\overline{\alpha_2}(f)(u_2) \wedge \overline{\alpha_3 \cdots \alpha_n}(f)(u'_2))\right] \\ &\vdots \\ &= \bigvee_{(u_1,u'_1)\in \mathbf{S}_x} \cdots \bigvee_{(u_{n-1},u'_{n-1})\in \mathbf{S}_{u'_{n-2}}} \left[\bigwedge_{i=1}^{n} \overline{\alpha_i}\,(f)(u_i)\right]. \end{aligned}$$

We note that if $|\alpha| = 2$, then there exist $u_1, u_2 \in S$ such that

$$\overline{\alpha}(f)(x) = \bigvee_{(u_1,u_2)\in \mathbf{S}_x} [\overline{\alpha_1}(f)(u_1) \wedge \overline{\alpha_2}(f)(u_2)].$$

As we know, the concept of fuzzy sets is a generalization of crisp sets. We can extend the concept of the α-ideal in ordered semigroups as follows.

Let $\mathbf{S}$ be an ordered semigroup and $\alpha \in \mathrm{F}$, where $|\alpha| = n$ for some $n \in \mathbb{N}$. We call a fuzzy set f of S a *fuzzy subsemigroup* of $\mathbf{S}$ if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$. A fuzzy subsemigroup f of $\mathbf{S}$ such that $x \leq y \Rightarrow f(x) \geq f(y)$ is said to be a *fuzzy α-ideal* of $\mathbf{S}$ if $f(x_1^n) \geq \bigwedge_{i\in 1(\alpha)} f(x_i)$ for all $x_1, \dots, x_n \in S$.

3. Main Results

In this section, we illustrate the interconnection of fuzzy α-ideals in ordered semigroups in terms of the reduction relation of full words. Moreover, we characterize fuzzy α-ideals in ordered semigroups using the multiplication over the set of all fuzzy sets defined on an ordered semigroup.

The following is an auxiliary result that helps examine our theorem.

Lemma 3.1. Let $\alpha, \beta \in \mathrm{B}$ such that $|\beta| = n$ for some $n \in \mathbb{N}$. If $\alpha \leq \beta$, then $1_n(\boldsymbol{u}) \subseteq 1(\beta)$ for some $\boldsymbol{u} \in \mathrm{C}_n(\alpha)$.

Proof. Assume that $\alpha \leq \beta$. Then, there exists $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathrm{C}_n(\alpha)$ such that $\boldsymbol{u} \leq_C \boldsymbol{v}$, where $\boldsymbol{v} = (v_1, \dots, v_n) \in \mathrm{C}_n(\beta)$. Let $i \in 1_n(\boldsymbol{u})$. Then, $u_i = 1$. Since $\boldsymbol{u} \leq_C \boldsymbol{v}$, $v_i = 1$. This implies that $i \in 1_n(\boldsymbol{v}) = 1(\beta)$. Therefore, we complete the proof. ■

By the above result, it is not difficult to verify that $1_n(\boldsymbol{u}) \subseteq 1(\beta)$ for all $\boldsymbol{u} \leq_C \boldsymbol{v}$, where $\boldsymbol{v} \in \mathrm{C}_n(\beta)$.

Theorem 3.2. Let $\mathbf{S}$ be an ordered semigroup, $\alpha, \beta \in \mathrm{F}$, and f a fuzzy set of S. Suppose that $\alpha \leq \beta$. We have that if f is a fuzzy α-ideal of $\mathbf{S}$, then it is a fuzzy β-ideal of $\mathbf{S}$.

Proof. Assume that f is a fuzzy α-ideal of $\mathbf{S}$. For the convenience, suppose that $|\alpha| = m$ and $|\beta| = n$ for some $m, n \in \mathbb{N}$. It is sufficient to illustrate that

$$f(x_1^n) \geq \bigwedge_{i \in 1(\beta)} f(x_i)$$

for all $x_1, \dots, x_n \in S$. Let $x_1, \dots, x_n \in S$. Since $\alpha \leq \beta$, there exists $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathrm{C}_n(\alpha)$ such that $1_n(\boldsymbol{u}) \subseteq 1(\beta)$. Suppose that $I = \{i_1, \dots, i_m\} \subseteq [n]$ be the set of all indices such that $u_i \neq \varepsilon$ for all $i \in I$. By this setting, we observe that $\alpha = u_{i_1} \cdots u_{i_m}$. We put

$$y_k = x_{i_k} \cdots x_{i_{k+1}-1}$$

for all $k \in [m]$. Then, for any $k \in [m]$ such that $u_{i_k} = 1$, we have that $y_k = x_{i_k}$. Thus,

$$\begin{aligned} f(x_1^n) &= f(y_1^m) \\ &= \bigwedge_{k \in 1(\alpha)} f(y_k) \\ &= \bigwedge_{k \in 1(\alpha)} f(x_{i_k}) \\ &= \bigwedge_{k \in 1_n(\boldsymbol{u})} f(x_i) \\ &\geq \bigwedge_{k \in 1(\beta)} f(x_i) \end{aligned}$$

Therefore, f is a fuzzy β-ideal of $\mathbf{S}$. ■

The following theorem characterizes fuzzy α-ideals in ordered semigroups in terms of the operation $\circ$.

Lemma 3.3 ([7]). Let $\mathbf{S}$ be an ordered semigroup and $f \in \mathrm{F}(\mathbf{S})$. Then, f is a fuzzy subsemigroup of $\mathbf{S}$ if and only if $f \circ f \subseteq f$.

The auxiliary result considered in [7] can be formulated into the concept of fuzzy α-ideals as follows.

Proposition 3.4. Let $\mathbf{S}$ be an ordered semigroup and $f_1, \dots, f_n \in \mathrm{F}(\mathbf{S})$. Then, we have that $(f_1 \circ \cdots \circ f_n)(x_1^n) \geq f_1(x_1) \wedge \cdots \wedge f_n(x_n)$ for all $x_1, \dots, x_n \in S$.

Theorem 3.5. Let $\mathbf{S}$ be an ordered semigroup, $f \in \mathrm{F}(\mathbf{S})$, and $\alpha = \alpha_1 \cdots \alpha_n \in \mathrm{F}$, where $|\alpha| = n$ for some $n \in \mathbb{N}$. Suppose that $x \leq y \Rightarrow f(x) \geq f(y)$. Then, the following statements are equivalent.

1. f is a fuzzy α-ideal of $\mathbf{S}$.
2. $f \circ f \subseteq f$ and $\overline{\alpha}(f) \subseteq f$.

Proof. $(1) \Rightarrow (2)$. Assume that f is a fuzzy α-ideal of $\mathbf{S}$. By Lemma 3.3, we have $f \circ f \subseteq f$. Let $x \in S$. If $\mathbf{S}_x = \emptyset$, then $\overline{\alpha}(f)(x) = 0 \leq f(x)$. Suppose that $\mathbf{S}_x \neq \emptyset$. By Remark 2.1, there exist $u_1, \dots u_n, u_1', \dots u_{n-2}' \in S$ such that

$$\begin{aligned} &\overline{\alpha}(f)(x) \\ &= \bigvee_{(u_1, u_1') \in \mathbf{S}_x} \cdots \bigvee_{(u_{n-1}, u_n) \in \mathbf{S}_{u_{n-2}'}} \left[\bigwedge_{i=1}^{n} \overline{\alpha_i}(f)(u_i)\right] \\ &= \bigvee_{(u_1^{n-1}, u_n) \in \mathbf{S}_x} \left[\bigwedge_{i \in 1(\alpha)} f(u_i)\right] \\ &\leq \bigvee_{(u_1^{n-1}, u_n) \in \mathbf{S}_x} f(u_1^n) \\ &\leq f(x). \end{aligned}$$

Therefore, we obtain our claim.

$(1) \Rightarrow (2)$. Assume that $f \circ f \subseteq f$ and $\overline{\alpha}(f) \subseteq f$. By Proposition 3.4 and $f \circ f \subseteq f$, we have that f is a fuzzy subsemigroup of $\mathbf{S}$. Finally, let $x_1, \ldots, x_n \in S$. Then,

$$\begin{aligned} f(x_1^n) &\geq \overline{\alpha}(f)(x_1^n) \\ &\geq \bigwedge_{i=1}^{n} \overline{\alpha_i}(f)(x_i) \\ &= \bigwedge_{i \in 1(\alpha)} f(x_i). \end{aligned}$$

Thus, we complete the proof. ■

4. Discussion

This paper introduces the concept of fuzzy α-ideals in ordered semigroups. This notion generalizes several fuzzy ideals, for example, fuzzy left (resp., right, bi-, interior, (m,n)-, n-interior) ideals. We obtain the interrelation among fuzzy α-ideals under the formation of the reduction relation. This result illustrates our complete understanding of their connections. Moreover, a complete characterization of fuzzy α-ideals is provided. For the future work, the readers may consider how fuzzy α-ideals share a connection to α-ideals in ordered semigroups.

5. Acknowledgement

The authors would like to thank the anonymous reviewers for their comments, which helped us improve the manuscript. The corresponding authors would like to thank the Faculty of Engineering, Rajamangala University of Technology Isan, Khon Kaen Campus, for providing partial financial support for the project.

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