# The connection between hyperideals and hybrid hyperideals through their level sets and characteristic functions in hypersemigroups

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## Abstract

While valuable in handling uncertainty, fuzzy and soft sets may sometimes prove inadequate for certain scenarios. By merging their respective strengths, leading to the advances of hybrid structures. These hybrid structures offer a robust framework for tackling uncertain problems effectively. In particular, they find application in studying hyperalgebraic structures, such as hypersemigroups. Traditionally, researchers have utilized hyperideals to explore the properties of hypersemigroups. However, our current study confirms this and provides a new perspective on the intricate relationship between hyperideals and hybrid hyperideals. By analyzing their level sets and characteristic functions, we clarify the connections between these two concepts and present an understanding of this field.

**Keywords:** hypersemigroups; hybrid hyperideals; hyperideals; hyperalgebraic systems

## 1. Introduction

Algebras serve as fundamental mathematical structures that are indispensable in addressing various problems in mathematics. Typically, the product of two elements of an algebra yields another element (see [1]). However, there are scenarios where traditional algebras fall short in adequately describing certain issues, such as the complications of blood type studies. To tackle such challenges, mathematicians have developed a specialized mathematical framework called hyperalgebras, introduced by Marty [2] in 1934. When combined with the property similar to adding numbers, hyperalgebras form hypersemigroups, also known as semihypergroups. Over the past two decades, the study of hypersemigroups has garnered significant attention within the mathematical community. Central to understanding the algebraic properties of hypersemigroups are the concepts of hyperideals. Numerous researchers have investigated various types of hyperideals in hypersemigroups, ascribing their algebraic properties and applications (see [3-7]).

The concepts of fuzzy sets and soft sets serve as invaluable mathematical tools in solving real-world problems fraught with uncertainty (see [8-9]). However, their efficacy is often constrained by their limited interpretative scope. Recognizing this, researchers have explored the synergistic potential of combining these two paradigms to enhance problem-solving capabilities. In 2018, Jun et al. [10] introduced hybrid structures, amalgamating fuzzy sets and soft sets to broaden their applicability. Their pioneering work extended to examining BCK-/BCI-algebras, shedding light on novel insights into these algebraic structures. Subsequently, numerous scholars have used hybrid structures to investigate many algebraic and hyperalgebraic structures, including semigroups and ordered semigroups (see [11-14]).

Hybrid structures appeared as a significant advancement in hypersemigroup theory in 2022, pioneered by Mekwian et al. [15] with their work on hybrid bi-hyperideals. Their contribution laid the foundation for understanding the relationships between bi-hyperideals and hybrid hyperideals in hyper-semigroups. Building upon this in 2023, the present authors further expanded the framework by introducing hybrid left and right hyperideals. They illustrated the connections among hybrid left, right, and bi-hyperideals, revealing distinct classes within these hybrid structures (see [16]).

In this paper, we extend the results established by Mekwian et al. in [15] by providing a generalized framework. Specifically, we characterize hybrid left (right, bi-) hyper-ideals through their level sets, thus offering a comprehensive characterization. Moreover, we demonstrate the inclusivity of the previous result. The paper is organized as follows: Section 1 delves into the historical context and the significance of hybrid structures within algebraic systems. Section 2 offers essential preliminaries to the groundwork for our subsequent discussions. Section 3 presents our primary contributions, describing hybrid left (right, bi-) hyperideals and their level sets.

## 2. Preliminaries

In this section, we recall some basic definitions of our study. A mapping $\circ: H \times H \to \mathcal{P}^*(H)$ on a nonempty set $H$ is said to be a *hyperoperation* on $H$, where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of $H$. A structure $\langle H;\circ\rangle$ is called a *hyperalgebra* if $\circ$ is a hyperoperation on $H$ (see [2]). A hyperoperation $\circ$ on $H$ gives rise to a binary operation $\hat{\circ}: \mathcal{P}(H) \times \mathcal{P}(H) \to \mathcal{P}(H)$ on $\mathcal{P}(H)$ defined by

$$A \mathbin{\hat{\circ}} B = \begin{cases} \bigcup\limits_{a\in A, b\in B} (a \circ b) & \text{if } A, B \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

for any $A, B \subseteq H$ where $\mathcal{P}(H)$ is the set of all subsets of $H$, (see [17]) A hyperalgebra $\langle H;\circ\rangle$ is said to be a *hypersemigroup* [18] if $A \mathbin{\hat{\circ}} (B \mathbin{\hat{\circ}} C) = (A \mathbin{\hat{\circ}} B) \mathbin{\hat{\circ}} C$ for all $A, B, C \subseteq H$. We usually denote a hypersemigroup $\langle H;\circ\rangle$ by the boldface letter $\mathbf{H}$ of its underlying set. Any product $A \mathbin{\hat{\circ}} B$ of any two subsets of $H$ will be denoted by $AB$, and for simplicity, we denote $\{x\}A$ and $A\{x\}$ by $xA$ and $Ax$, respectively. Moreover, the $n$-product $A \mathbin{\hat{\circ}} \cdots \mathbin{\hat{\circ}} A$ is written as $A^n$, where $n$ is a natural number.

Let $\mathbf{H}$ be a hypersemigroup. A nonempty subset $A$ of $H$ is said to be a *left (right) hyperideal* [19] of $\mathbf{H}$ if $HA \subseteq A$ ($AH \subseteq A$) and a *bi-hyperideal* [20] of $\mathbf{H}$ if $AA \subseteq A$

and $AHA \subseteq A$. It is not difficult to see that any left (right) hyperideal is a bi-hyperideal.

A mapping $f: X \to [0,1]$ from a non-empty set $X$ to a closed unit interval is called a *fuzzy set* [9] in $X$. Let $U$ and $E$ be nonempty sets. A *soft set* $f$ in $E$ over $U$ is a mapping $f: E \to \mathcal{P}(U)$. Combining such two concepts, we obtain the concept of hybrid structures defined by the following definition.

**Definition 2.1** ([10]). *Let $E$ and $U$ be non-empty sets. A mapping $f: E \to \mathcal{P}(U) \times [0,1]$ is called a* hybrid structure *in $E$ over $U$.*

Let $f$ be a hybrid structure in $E$ over $U$. We can see that $f$ can be considered as a pair $f := (f_1, f_2)$ of a soft set $f_1$ in $E$ over $U$ and a fuzzy set $f_2$ in $E$. From now on, we let $U$ be an arbitrary nonempty set. For any $(T,t) \in \mathcal{P}(E) \times [0,1]$ can be regarded as a constant hybrid structure in $E$ over $U$ defined by $T(x) = T$ and $t(x) = t$ for all $x \in E$. Let $A$ be a subset of $E$. The *characteristic hybrid structure* $\chi_A$ in $E$ over $U$ is defined by $\chi_A(x) := (U, 0)$ if $x \in A$ and $\chi_A(x) := (\emptyset, 1)$ if $x \notin A$. For any hybrid structures $f$ and $g$ in $E$ over $U$, we let $f \ll g$ if and only if $f_1(x) \subseteq g_1(x)$ and $f_2(x) \geq g_2(x)$ for all $x \in E$. If $g \gg f$, we mean $f \ll g$.

The concept of hybrid structures can be used to study the algebraic properties of hypersemigroups through the following notions. Let $\mathbf{H}$ be a hypersemigroup and $f$ a hybrid structure in $H$ over $U$. Then, $f$ is said to be:

(1) a *hybrid subhypersemigroup* in $\mathbf{H}$ over $U$ if $f_1(x) \cap f_1(y) \subseteq \bigcap_{u \in xy}\{f_1(u)\}$ and $f_2(x) \vee f_2(y) \geq \bigvee_{u \in xy}\{f_2(u)\}$ for all $x, y \in H$;

(2) a *hybrid left hyperideal* in $\mathbf{H}$ over $U$ if $f_1(y) \subseteq \bigcap_{u \in xy}\{f_1(u)\}$ and $f_2(y) \geq \bigvee_{u \in xy}\{f_2(u)\}$ for all $x, y \in H$;

(3) a *hybrid right hyperideal* in $\mathbf{H}$ over $U$ if $f_1(x) \subseteq \bigcap_{u \in xy}\{f_1(u)\}$ and $f_2(x) \geq \bigvee_{u \in xy}\{f_2(u)\}$ for all $x, y \in H$;

(4) a *hybrid bi-hyperideal* in $\mathbf{H}$ over $U$ if $f$ is a hybrid subhypersemigroup such that $f_1(x) \cap f_1(z) \subseteq \bigcap_{u \in xyz}\{f_1(u)\}$ and $f_2(x) \vee f_2(z) \geq \bigvee_{u \in xyz}\{f_2(u)\}$ for all $x, y, z \in H$.

In [15], the authors illustrated a connection between bi-hyperideals and hybrid bi-hyper-ideals in hypersemigroups using the notion of characteristic hybrid structures.

**Theorem 2.2.** *Let $\mathbf{H}$ be a hypersemigroup, and $A$ a nonempty subset of $H$. The following statements are equivalent.*

(1) *$A$ is a bi-hyperideal of $\mathbf{H}$.*

(2) *$\chi_A$ is a hybrid bi-hyperideal in $\mathbf{H}$ over $U$.*

## 3. Main Results

In this section, we generalize Theorem 2.2 by means of level sets of hybrid structures. Let $\mathbf{H}$ be a hypersemigroup, $f$ a hybrid structure in $H$ over $U$, $T \subseteq H$, and $t \in [0,1]$. We define a *lever set* of $f$ by $\text{lev}(f; (T,t)) := \{x \in H : f(x) \gg (T,t)\}$.

We begin our main result by characterizing hybrid left (right) hyperideals in hypersemigroup using level sets.

**Theorem 3.1.** *Let $\mathbf{H}$ be a hypersemigroup, and $f$ a hybrid structure in $H$ over $U$. Then, the following statements are equivalent.*

(1) *$f$ is a hybrid left (right) hyperideal in $\mathbf{H}$ over $U$.*

(2) *the nonempty level set $\text{lev}(f; (T,t))$ is a left (right) hyperideal of $\mathbf{H}$ for any*

$\emptyset \neq T \subseteq H$ *and* $t \in [0,1)$.

*Proof.* We prove only the case of hybrid left hyperideals. For another case can be proved similarly.

$(1) \Rightarrow (2)$. Let $T$ be a nonempty subset of $H$ and $t \in [0,1)$. Let $x \in H$ and $y \in \mathrm{lev}(f;(T,t))$. Then, $f_1(y) \supseteq T$ and $f_2(y) \leq t$. By our assumption, we have $f_1(xy) \supseteq \bigcap_{u \in xy}\{f_1(u)\} \supseteq f_1(y) \supseteq T$ and $f_2(xy) \leq \bigvee_{u \in xy}\{f_2(u)\} \leq f_2(y) \leq t$. This means that $f(xy) \gg (T,t)$. That is, $xy \in \mathrm{lev}(f;(T,t))$. Therefore, $\mathrm{lev}(f;(T,t))$ is a left hyperideal of $\mathbf{H}$.

$(2) \Rightarrow (1)$. Let $x, y \in H$. If $f(y) \ll (\emptyset, 1)$, then $f_1(y) = \emptyset$ and $f_2(y) = 1$. This implies that $f_1(y) \subseteq \bigcap_{u \in xy}\{f_1(u)\}$ and $f_2(y) \geq \bigvee_{u \in xy}\{f_2(u)\}$. Suppose that $f_1(y) \neq \emptyset$ and $f_2(y) < 1$. It is clear that $y \in \mathrm{lev}(f;f(y))$. By the left hyperideality of $\mathrm{lev}(f;f(y))$, we have $xy \in \mathrm{lev}(f;f(y))$. This implies that $f_1(v) \supseteq f_1(y)$ and $f_2(v) \leq f_2(y)$ for all $v \in xy$. That is, $\bigcap_{u \in xy} f_1(u) \supseteq f_1(y)$ and $\bigvee_{u \in xy} f_2(u) \leq f_2(y)$. This shows that $f$ is a hybrid left hyperideal in $\mathbf{H}$ over $U$.

Our last main result illustrates a characterization of hybrid bi-hyperideals.

**Theorem 3.2.** *Let* $\mathbf{H}$ *be a hypersemigroup, and* $f$ *a hybrid structure in* $H$ *over* $U$*. Then, the following statements are equivalent.*

(1) $f$ *is a hybrid bi-hyperideal in* $\mathbf{H}$ *over* $U$*.*

(2) *the nonempty level set* $\mathrm{lev}(f;(T,t))$ *is a bi-hyperideal of* $\mathbf{H}$ *for any* $\emptyset \neq T \subseteq H$ *and* $t \in [0,1)$.

*Proof.* $(1) \Rightarrow (2)$. Let $T$ be a nonempty subset of $H$ and $t \in [0,1)$. First we show that $\mathrm{lev}(f;(T,t))$ is a subhypersemigroup. Let $x, y \in \mathrm{lev}(f;(T,t))$. Then, $f(x) \gg (T,t)$ and $f(y) \gg (T,t)$. Hence, $f_1(xy) \supseteq \bigcap_{u \in xy}\{f_1(u)\} \supseteq f_1(x) \cap f_1(y) \supseteq T \cap T = T$ and $f_2(xy) \leq \bigvee_{u \in xy}\{f_2(u)\} \leq f_2(x) \vee f_2(y) \leq t \vee t = t$. This means that $\mathrm{lev}(f;(T,t))$ is a subhypersemigroup. Let $y \in H$ and $x, z \in \mathrm{lev}(f;(T,t))$. Then, $f_1(x) \supseteq T, f_1(z) \supseteq T, f_2(x) \leq t$, and $f_2(z) \leq t$. By our presumption, we have $T \subseteq f_1(x) \cap f_1(z) \subseteq \bigcap_{u \in xyz}\{f_1(u)\} \subseteq f_1(xyz)$ and $t \geq f_2(x) \vee f_2(z) \geq \bigvee_{u \in xyz}\{f_2(u)\} \geq f_2(xyz)$. This means that $f(xyz) \gg (T,t)$. That is, $xyz \in \mathrm{lev}(f;(T,t))$. Altogether, $\mathrm{lev}(f;(T,t))$ is a bi-hyperideal of $\mathbf{H}$.

$(2) \Rightarrow (1)$. Let $x, y, z \in H$. If $(f_1(x) \cap f_1(z), f_2(x) \vee f_2(z)) \ll (\emptyset, 1)$, then $f_1(x) \cap f_1(z) = \emptyset \subseteq \bigcap_{u \in xyz}\{f_1(u)\}$ and $f_2(x) \vee f_2(z) = 1 \geq \bigvee_{u \in xyz}\{f_2(u)\}$. Now, suppose that $f_1(x) \cap f_1(z) \neq \emptyset$ and $f_2(x) \vee f_2(z) < 1$. Put $T := f_1(x) \cap f_1(z)$ and $t := f_2(x) \vee f_2(z)$. Then, it is not difficult to see that $x, z \in \mathrm{lev}(f;(T,t))$. Since $\mathrm{lev}(f;(T,t))$ is a bi-hyperideal of $\mathbf{H}$, $x, y, z \subseteq \mathrm{lev}(f;(T,t))$. Therefore, $f_1(v) \supseteq f_1(x) \cap f_1(z)$ and $f_2(v) \leq f_2(x) \vee f_2(z)$ for all $v \in xyz$. This means that $\bigcap_{u \in xyz}\{f_1(u)\} \supseteq f_1(x) \cap f_1(z)$ and $\bigvee_{u \in xyz}\{f_2(u)\} \leq f_2(x) \vee f_2(z)$. Illustrating that $f$ is a hybrid subhypersemigroup in $\mathbf{H}$ over $U$ can be proved similarly. Hence, $f$ is a hybrid bi-hyperideal in $\mathbf{H}$ over $U$.

The above theorem can be reduced regarding the characteristic hybrid structures as follows.

**Corollary 3.3.** *Let* $\mathbf{H}$ *be a hypersemigroup, and* $A$ *a nonempty subset of* $H$*. Then, the following statements are equivalent.*

*(1)* $\chi_A$ *is a hybrid left (right, bi-) hyperideal in* $\mathbf{H}$ *over* $U$*.*

*(2) A is a left (right, bi-) hyperideal of* $\mathbf{H}$.

*Proof.* Since $\text{lev}(\chi_A; (T,t)) = \{x \in H: \chi_A(x) \gg (T,t)\} = A$, where $\emptyset \neq T \subseteq H$ and $t \in [0,1)$, we obtain the proof.

## 4. Conclusion

The current paper provides the connections between left (right, bi-) hyperideals and hybrid left (right, bi-) hyperideals by level sets. These generalize these connections given by the characteristic functions.

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