On a class of semisimple ordered semigroups

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Abstract

If one examines a situation where globally idempotent ordered semigroups contain maximal one-sided ideals, numerous instances of semisimple ordered semigroups can be identified. In this paper, we find some sufficient conditions for a globally idempotent ordered semigroups to be semisimple ordered semigroups.

Keywords: ordered semigroup; semisimple; globally idempotent; regular; prime ideal.

1. Introduction

The study of semigroups whose ideals are representable as a product or an intersection of prime or maximal ideals has interested Dorofeeva, Grillet, Mennepalli, Petrich, Szssz and Satyanarayana. It is observed that semisimple semigroups can easily be characterized as those semigroups whose ideals are intersections of prime ideals. Among globally idempotent semigroups containing maximal one-sided ideals one, one can find many examples of semisimple semigroups. In [1], Satyanarayana finds some sufficient conditions for a globally idempotent semigroups to be semisimple semigroups. In this paper, the results are a generalization of Satyanarayana in [1].

2. Preliminaries

A semigroup $(S,\cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that for any x,y,z in S, $x\leq y$ implies $zx\leq zy$ and $xz\leq yz$, is called a *partially ordered semigroup,* or simply an *ordered semigroup*. Under the trivial relation, $x\leq y$ if and only if $x=y$, it is observed that every semigroup is an ordered semigroup.

Let $(S,\cdot,\leq)$ be an ordered semigroup. For two nonempty subsets A,B of S, we write AB for the set of all elements xy in S where $x\in A$ and $y\in B$, and write $(A]$ for the set of all elements x in S such that $x\leq a$ for some a in A, i.e.,

$$(A]=\left\{x\in S \middle| x\leq a \quad \exists a\in A\right\}.$$

In particular, we write Ax for $A\{x\}$, and xA for $\{x\}A$. It was shown in [2] that the following hold:

(1) $A \subseteq (A]$ and $((A]] = (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A](B]] = (A(B]] = ((A]B]$
$= (AB]$;

(4) $(A](B] \subseteq (AB]$;

(5) $(A]B \subseteq (AB]$ and $A(B] \subseteq (AB]$;

(6) If $\{A_k\}_{k\in K}$ is a family of nonempty subsets of S, then
$(\bigcup_{k\in K} A_k] = \bigcup_{k\in K} (A_k]$ and
$(\bigcap_{k\in K} A_k] \subseteq \bigcap_{k\in K} (A_k]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *left (resp., right) ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp., $AS \subseteq A$);

(ii) $(A] = A$, that is, for any x in A and y in S, $y \leq x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. It is known that the union or intersection of two ideals of S is an ideal of S.

An element a of an ordered semigroup $(S, \cdot, \leq)$, the *principal left(resp., right, two-sided) ideal* generated by a is of the form $L(a) = (a \cup Sa]$ (resp., $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$).

Let $(S, \cdot, \leq)$ be an ordered semigroup. A left ideal A of S is said to be *proper* if $A \subset S$. A proper right and two-sided ideals are defined similarly. S is said to be *left (resp., right) simple* if S does not contain proper left (resp., right) ideals. S is said to be *simple* if S does not contain proper ideals.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup T of S is called *left simple(resp. right simple)* if T is the only left ideal(resp. right ideal) of T; it is called *simple* if it is the only ideal of T. A subsemigroup T of S is left simple if and only if $(Ta]_T = T$ for every $a \in T$. T is right simple if and only if $(aT]_T = T$ for every $a \in T$, T is simple if and only if $(TaT]_T = T$ for every $a \in T$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A proper left (resp., right) ideal A of S is said to be *maximal left(resp., maximal right)* if for any left (resp., right) ideal B of S, if $A \subset B \subseteq S$, then $B = S$. A proper ideal A of S is said to be *maximal* if for any ideal B of S, if $A \subset B \subseteq S$, then $B = S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal I of S is said to be *prime* if for any ideals A, B of S, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ideal I of S is said to be *completely prime* if for any $a, b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$. An ideal I of S is said to be *semiprime* if for any ideal A of S, $A^2 \subseteq I$ implies $A \subseteq I$. An ideal I of S is said to be *completely semiprime* if for any $a \in S$, $a^n \in I$ for any positive integer n implies $a \in I$ (see [3]). An ordered semigroup $(S, \cdot, \leq)$ is called a *globally idempotent* if $S = (S^2]$.

Let $(S,\cdot,\leq)$ be an ordered semigroup. The intersection of all ideals of S, if it is nonempty, is called a *kernel* of S, and it will be denoted by $K(S)$. The intersection of all prime ideals of S will be denoted by Q^*. The intersection of all maximal left ideals of S will be denoted by L^*. And the intersection of all maximal right ideals of S will be denoted by R^*.

An ideal A of an ordered semigroup $(S,\cdot,\leq)$, the intersection of all prime ideals of S containing A, will be denoted by $Q^*(A)$.

An element a of an ordered semigroup $(S,\cdot,\leq)$ is called a *semisimple element* in S if $a\in(SaSaS]$. And S is said to be *semisimple* if every element of S is semisimple (see [3]).

An element a of an ordered semigroup $(S,\cdot,\leq)$ is said to be *left regular (resp., right regular, regular, intra-regular)* if there exists x,y in S such that $a\leq xa^2$ (resp., $a\leq a^2x$, $a\leq axa$, $a\leq xa^2y$) (see [3]). It is observed that left regular elements, right regular elements, regular elements, and intra-regular elements are all semisimple.

An element e of an ordered semigroup $(S,\cdot,\leq)$ is called an *identity element* of S if $ex=x=xe$ for any $x\in S$. The *zero element* of S, defined by Birkhoff, is an element 0 of S such that $0\leq x$ and $0x=0=x0$ for all $x\in S$. S is said to be 0-*simple* if $S^2\neq\{0\}$ and $\{0\}$ is the only proper ideal of S (see [4]).

Let $(S,\cdot,\leq_S)$ and $(T,*,\leq_T)$ be ordered semigroups, $f:S\longrightarrow T$ a mapping from S into T. The mapping f is called *isotone* if $x,y\in S$, $x\leq_S y$ implies $f(x)\leq_T f(y)$ and *reverse isotone* if $x,y\in S$, $f(x)\leq_T f(y)$ implies $x\leq_S y$. The mapping f is called a *homomorphism* if it is isotone and satisfies $f(xy)=f(x)*f(y)$ for all $x,y\in S$. The mapping f is called a *isomorphism* if it is reverse isotone onto homomorphism. The ordered semigroups S and T are called *isomorphic,* in symbols $S\cong T$ if there exists an isomorphism between them.

An ordered semigroup V is called an *ideal extension* (or just an *extension*) of an ordered semigroup S by an ordered semigroup Q, if Q has a zero 0, $S\cap(Q\setminus\{0\})=\varnothing$, and there exists an ideal K of V such that $K\cong S$ and $V/K\cong Q$ (see [5]).

Let $(S,\cdot,\leq)$ be an ordered semigroup and K an ideal of S. S/K is called the *Rees quotient ordered semigroup* of S, where 0 is an arbitrary element of K. It is observed that $K\cap[(S/K)\setminus\{0\}]=\varnothing$, $K\cong K$ and $S/K\cong S/K$ under the identity mapping and so S is an ideal extension of K by S/K.

3. Main results

We begin this section with the following lemma.

Lemma 3.1 [6] Let $(S,\cdot,\leq)$ be an ordered semigroup. The following statements are equivalent:

(1) S is semisimple;

(2) $(A^2]=A$ for any ideal A of S;

(3) $A\cap B=(AB]$ for any ideal A,B of S;

(4) $I(a)\cap I(b)=(I(a)I(b)]$ for any $a,b\in S$;

(5) $(I(a)^2]=I(a)$ for any $a\in S$.

Lemma 3.2 [6] Let $(S,\cdot,\leq)$ be an ordered semigroup. Then S is semisimple if and only if every ideal of S is semiprime.

Lemma 3.3 [6] Let $(S,\cdot,\leq)$ be an ordered semigroup and A an ideal of S. Then A is semiprime if and only if $Q^*(A)=A$.

Proposition 3.4 Let $(S,\cdot,\leq)$ be an ordered semigroup. The following statements are equivalent:

(1) Every proper ideal of S is an intersection of prime ideals of S;

(2) $(A^2]=A$ for any ideal A of S;

(3) $A\cap B=(AB]$ for any ideal A,B of S;

(4) $I(a)\cap I(b)=(I(a)I(b)]$ for any $a,b\in S$;

(5) $(I(a)^2]=I(a)$ for any $a\in S$;

(6) S is semisimple.

Proof. We have (2) to (6) are equivalent by Lemma 3.1.

$(1)\longrightarrow(2)$. Let A be an ideal of S. Then $(A^2]=\bigcap_i Q_i$, where each Q_i is prime ideal of S. We have $A^2\subseteq(A^2]=\bigcap_i Q_i$, for all i. This implies $A\subseteq Q_i$ for all i. Thus $A\subseteq\bigcap_i Q_i=(A^2]$ and so $(A^2]=A$.

$(6)\longrightarrow(1)$. Let A be proper ideal of S. Since S is semisimple, we have A is semiprime by Lemma 3.2. Thus $Q^*(A)=A$ by Lemma 3.3.

Proposition 3.5 Let $(S,\cdot,\leq)$ be an ordered semigroup. Then every right ideal of S is an intersection of prime ideals of S if and only if every right ideal of S is two-sided and for every $a\in S$, $a\in(aSaS]$.

Proof. Assume that every right ideal of S is an intersection of prime ideals of S. Let A be a right ideal of S. Then $A=\bigcap_i Q_i$, where each Q_i is prime ideal of S. We have $SA=S(\bigcap_i Q_i)\subseteq\bigcap_i Q_i=A$. Thus A is a two-sided ideal of S. Let $a\in S$. Then $a\in(SaSaS]\subseteq(aSaS]$ by Proposition 3.4. The converse is evident from Proposition 3.4.

It is easy to see the following proposition:

Proposition 3.6 Let $(S,\cdot,\leq)$ be an ordered semigroup. Then every right ideal of S is prime if and only if $(R(x)R(y)]=R(x)$ or $(R(x)R(y)]=R(y)$ for every $x,y\in S$.

Similarly, we prove the following:

Proposition 3.7 Let $(S,\cdot,\leq)$ be an ordered semigroup. Then every ideal of S is prime if and only if $(I(x)I(y)]=I(x)$ or $(I(x)I(y)]=I(y)$ for every $x,y\in S$.

Lemma 3.8 Let $(S,\cdot,\leq)$ be a left cancellative ordered semigroup. If S is a right simple, then S is regular.

Proof. Let $a\in S$. Since S is a right simple, $a\leq ax$ for some $x\in S$. We have $a^2\leq axa$. Since S is a left cancellative, $a\leq xa$. Since $x\in(aS]$, $x\leq ay$ for some $y\in S$. It follows that $a\leq xa\leq aya\in aSa$. Thus S is regular.

Theorem 3.9 Let $(S,\cdot,\leq)$ be an ordered semigroup in which every right ideal of S is intersection prime ideals of S. Then S is completely regular if S is a left cancellative and left simple.

Proof. Let $a\in S$. Then $a\in(aSaS]=(aS(aS]]\subseteq(a^2S]$ by Proposition 3.5. Thus S is right regular. We have $a\in(aSaS]\subseteq(aS]$ Then $a\leq ax$ for some $x\in S$. It follows that $a^2\leq axa$. Since S is a left cancellative, $a\leq xa$. Since S is left simple, $x\leq ya$ for some $y\in S$. Thus $a\leq xa\leq ya^2\in Sa^2$, that is, S is left regular. Now we claim that S is right simple. If S is not right simple. Then S has a unique maximal right ideal M, which is the union of all proper right ideals of S. Let $m\in M$ and $s\in S$. Then $m\leq mymz$ for some $y,z\in S$ by Proposition 3.5. We have $ms\leq mymzs$. Since S is a left cancellative, $s\leq ymzs\in SMS\subseteq M$. It follows that $S=M$ This is a contradiction. Thus S right simple. We have S is regular by Lemma 3.8. Thus S is completely regular.

Proposition 3.10 If $(S,\cdot,\leq)$ is an ordered semigroup with $S=(S^2]$, then every maximal ideal of S is prime.

Proof. Let M be a maximal ideal of S. Denote $S\setminus M=P$. We claim that $P\subseteq(P^2]$. Since M is a maximal, $S=M\cup P$. We have $S=(S^2]=((M\cup P)^2]\subseteq M\cup(P^2]$. Thus $P\subseteq(P^2]$. Let A,B be an ideal of S such that $AB\subseteq M$. If M is not prime. Since M is a maximal, $S=M\cup A$ and $S=M\cup B$. Then $P\subseteq A$ and $P\subseteq B$. It follows that $P\subseteq(P^2]\subseteq(AB]\subseteq M$. This is a contradiction. Thus M is prime.

Definition 3.11 An ordered semigroup $(S,\cdot,\leq)$ is called *intersective* if every proper ideal of S is an intersection of maximal ideals of S. A 0-simple ordered semigroup is a trivial example of an intersective ordered semigroup.

It is easy to see the following proposition:

Proposition 3.12 Let $(S,\cdot,\leq)$ be an intersective ordered semigroup. Then S is semisimple if and only if $S=(S^2]$.

Remark 3.13 Every semisimple ordered semigroups in which proper prime ideals are maximal, are intersective.

Proposition 3.14 If $(S,\cdot,\leq)$ is intersective semisimple ordered semigroup, then every proper prime ideal of S is maximal.

Proof. Let A be a proper prime ideal of S. Since S is intersective, $A=\bigcap_i M_i$, where each M_i is maximal ideal of S. Since S is semisimple, we have

$A = \cap_i M_i = (M_1 M_2 \cdots M_i]$. Since A is prime, $M_\alpha \subseteq A$ for some α. Thus $A = M_\alpha$.

Lemma 3.15 Let $(S, \cdot, \leq)$ be a globally idempotent ordered semigroup containing maximal right ideals. If $R^* = \varnothing$, then for any $a \in S$, $a \in (aSaS]$ or there exists a maximal right ideal M of S not containing a such that

(1) $(SM] = S$;

(2) $S = (M^n] \cup (aS]$ for every positive integer n and

(3) $S = M^\infty \cup (aS]$, where $M^\infty = \cap_{n=1}^{\infty} (M^n]$.

Proof. Let $a \in S$. Since $R^* = \varnothing$, there exists a maximal right ideal M not containing a. Then $S = M \cup (a \cup aS]$. We have

$$S = (S^2] = ((M \cup (a \cup aS])S] \subseteq M \cup (aS].$$

Thus $S = M \cup (aS]$. This implies $S = (SM] \cup (SaS]$. If $a \notin (SM]$, then $a \leq xay$ for some $x, y \in S$. Since $a \notin M$, we have $x \notin M$. It follows that $x \leq az$ for some $z \in S$. Thus $a \leq xay \leq azay \in aSaS$. If $a \in (SM]$, then $S = (SM]$. We have

$$S = (SM] = ((M \cup (aS])M] = (M^2] \cup (a(SM]] = (M^2] \cup (aS]$$

and

$$S = (M \cup (aS]) \cap ((M^2] \cup (aS]) \subseteq (M \cap (M^2]) \cup (aS].$$

Proceeding in the way we obtain $S = (M^n] \cup (aS]$ for every positive integer n and $S = M^\infty \cup (aS]$.

Theorem 3.16 Let $(S, \cdot, \leq)$ be a globally idempotent ordered semigroup containing maximal right ideals. If $R^* = \varnothing$, then S is semisimple if one of the following conditions is satisfied:

(1) $S \neq (SM]$ for every maximal right ideal M of S;

(2) $S \neq (M^n] \cup (aS]$ for every maximal right ideal M of S not containing a;

(3) $M^\infty = \varnothing$ for every maximal right ideal M of S.

Proof. If S satisfies the following conditions (1) and (2), then S is semisimple follow from Lemma 3.15. Let S satisfies the following condition (3) and $a \in S$. Then $a \in (aSaS]$ or there exists a maximal right ideal M of S not containing a such that $S = M^\infty \cup (aS]$ by Lemma 3.15. If $a \in (aSaS]$, then obviously S is semisimple. If there exists a maximal right ideal M not containing a such that $S = M^\infty \cup (aS]$. Then $S = (aS]$. We have

$$S = (aS] = (a^2 S] = (a^4 S] \subseteq (SaSaS].$$

Thus S is semisimple.

Theorem 3.17 Let $(S, \cdot, \leq)$ be a globally idempotent ordered semigroup containing maximal right ideals. If $R^* = \varnothing$, then S is regular if either one of the following conditions is satisfied:

(1) $L^* = \varnothing$ and S contains maximal left ideals;

(2) S is left simple.

Proof. Let $a \in S$. Since $R^* = \varnothing$, there exists a maximal right ideal M of S not containing a. Then $S = M \cup (a \cup aS]$. We have

$$S = (S^2] = ((M \cup (a \cup aS])S] \subseteq M \cup (aS].$$

Thus $S = M \cup (aS]$. Similarly, if $L^* = \varnothing$ and S contains maximal left ideals, then there exists a maximal left ideal N of S not containing a such that $S = N \cup (aS]$. It follows that $a \in (aS]$ and $a \in (Sa]$. This implies $(aS] = S = (Sa]$. Thus $a \in (aS] = (a(Sa]] = (aSa]$ and so S is regular. In the second case if S is left simple. Then $S = (Sa]$. Since $a \notin M$, $a \in (aS]$. This implies S is regular.

Theorem 3.17 Let $(S, \cdot, \leq)$ be a semisimple ordered semigroup containing maximal right and left ideals. If every prime ideal of S is a maximal right ideal as well as a maximal left ideal, then S is an ideal extension of a simple subsemigroup by a regular ordered semigroup.

Proof. Let $x \in L^* \cup R^*$.

If $x \notin K(S)$. Since S is semisimple, $K(S) = Q^*$. by Theorem 2.8 in [3]. Then there exists prime ideal of S not containing x. We have $x \notin L^* \cup R^*$ by the hypothesis. This is a contradiction. Thus $L^* \cup R^* \subseteq K(S)$. Let $x \in S / K(S)$. If $x \notin K(S)$, then $x \notin L^* \cap R^*$. Since S is semisimple, $S = (S^2]$. It follows that x is regular as above. We have $K(S)$ is a simple subsemigroup by Lemma 3.2 in [7]. Clearly S is an ideal extension of $K(S)$ by $S / K(S)$. This complete the proof.

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